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Abstract

We develop the theory of Abelian functions defined using a tetragonal curve of genus six, discussing in detail the cyclic curve $y^4 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$. We construct Abelian functions using the multivariate σ-function associated with the curve, generalizing the theory of the Weierstrass $\wp$-function. We demonstrate that such functions can give a solution to the KP-equation, outlining how a general class of solutions could be generated using a wider class of curves. We also present the associated partial differential equations satisfied by the functions, the solution of the Jacobi inversion problem, a power series expansion for $\sigma(\boldsymbol{u})$ and a new addition formula.

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1. Introduction

Recent times have seen a revival of interest in the theory of Abelian (multiply periodic) functions associated with algebraic curves. This topic can be dated back to the Weierstrass theory of elliptic functions which we use as a model. Let $\sigma(u)$ and $\wp(u)$ be the standard Weierstrass functions (see for example [1]). The $\wp$-function can be used to parametrize an elliptic curve $y^2 = 4x^3 - g_2 x - g_3$, and satisfies the following well-known formulae:

$$\wp(u) = -\frac{\mathrm{d}^2}{\mathrm{d}u^2} \log \sigma(u), \tag{1}$$

$$(\wp'(u))^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \tag{2}$$

$$\wp''(u) = 6\wp(u)^2 - \frac{1}{2}g_2. \tag{3}$$

The σ-function satisfies a power series expansion,

$$\sigma(u) = u - \frac{1}{240}g_2 u^5 - \frac{1}{840}g_3 u^7 - \frac{1}{1\,61\,280}g_2^2 u^9 - \frac{1}{22\,17\,600}g_2 g_3 u^{11} + \cdots, \tag{4}$$

and a two-term addition formula,

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v). \tag{5}$$

Taking logarithmic derivatives of this will give the standard addition formula for $\wp(u)$. This paper will generalize equations (1)–(5) for a previously unconsidered class of functions.

The study of Abelian functions associated with the simplest hyperelliptic curves (those of genus two) goes back to the start of the 20th century. Klein's generalization of the Weierstrass theory is described in Baker's classic texts [2, 3], while Buchstaber *et al* [4] give a more recent study of the general hyperelliptic case. Further generalization has been structured by considering (with the notation of [5]) classes of (*n, s*)-*curves*. These are curves with the equation

$$y^n - x^s - \sum_{\alpha,\beta} \mu_{[ns-\alpha n-\beta s]} x^\alpha y^\beta \qquad \mu_j \text{ constants}, \tag{6}$$

where $\alpha, \beta \in \mathbb{Z}$ with $\alpha \in (0, s-1)$, $\beta \in (0, n-1)$ and $\alpha n + \beta s < ns$. The cyclic subset of such a class of curves is generated by setting $\beta = 0$. We suppose that (n, s) are coprime, in which case the curves have genus $g = \frac{1}{2}(n-1)(s-1)$ and a unique branch point ∞ at infinity.

In the last few years a good deal of progress has been made on the theory of Abelian functions associated with trigonal curves (those with $n = 3$). The σ-function realization of these functions was developed first in [6, 7], with the two canonical cases studied in detail in [8, 9].

In this paper we consider the next logical class (those with $n = 4$). These are the *tetragonal curves,* and we have started by looking at the curves of lowest genus and simplified by considering the cyclic subclass. We construct the multivariate σ-function associated with this curve and use it to define and analyse classes of Abelian functions, generalizing the theory of the Weierstrass $\wp$-function. A key component of our work is the construction of a series expansion for the σ-function. This technique was first developed for the trigonal case in [10]; however, the computation involved for the present expansion is significantly greater. The latter computations are performed in parallel with the use of the Distributed Maple software (see [11, 12]).

The applications of Abelian functions to integrable systems and soliton theory have been the topic of research for some time (see for example [13, 14]). It is well known that the elliptic $\wp$-function could be used to construct a solution to the KdV equation. Similar solutions to nonlinear equations have been derived from higher genus curves, for example in [7] where the function $\wp_{33}$ associated with the (3,4)-curve was shown to be a solution of the Boussinesq equation. This has suggested a more general link between such functions and the integrable KP hierarchy. We demonstrate how the Abelian functions we define can give a solution to the KP-equation, outlining how similar solutions will also be found from any (4, *s*)-curve.

This paper is organized as follows. We give the basic properties of the curve we consider in section 2, including explicit constructions of the differentials on the curve and a set of weights that render the key equations homogeneous. Then in sections 3 and 4 we define the σ-function and Abelian functions associated with this curve. Section 5 discusses a key theorem satisfied by the $\wp$-functions which we use to give a solution to the Jacobi inversion problem. In section 6 we derive some properties of the σ-function including the series expansion, while in section 7 we use this to generate relations between the Abelian functions. Section 8 demonstrates how solutions to the KP-equation can be constructed from Abelian functions. Finally in section 9 we give the derivation of a two-term addition formula.

2. The purely tetragonal curves

We will investigate Abelian functions associated with a tetragonal curve. The simplest general tetragonal curve is, in the notation of the (n, s)-curves, a $(4, 5)$-curve. It is given by $g(x, y) = 0$ where

$$
\begin{aligned}
g(x, y) = {} & y^4 + (\mu_1 x + \mu_5)y^3 + (\mu_2 x^2 + \mu_6 x + \mu_{10})y^2 \\
& + (\mu_3 x^3 + \mu_7 x^2 + \mu_{11} x + \mu_{15})y \\
& - (x^5 + \mu_4 x^4 + \mu_8 x^3 + \mu_{12} x^2 + \mu_{16} x + \mu_{20}) \qquad (\mu_j \text{ constants}).
\end{aligned}
$$

In this paper we further simplify by considering the *cyclic subclass* of this family. These are the curves C, given by

$$
C: \quad f(x, y) = 0 \qquad (\lambda_j \text{ constants})
$$

where

$$
f(x, y) = y^4 - (x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0). \tag{7}
$$

The curve C has genus $g = 6$, the unique branch point ∞ at infinity and is referred to as the *purely tetragonal* or *strictly tetragonal* curve. It contains an extra level of symmetry demonstrated by the fact that it is invariant under

$$
[\zeta]: \quad (x, y) \rightarrow (x, \zeta y), \tag{8}
$$

where ζ is a fourth root of unity.

For any (n, s)-curve we can define a set of weights for the variables of the theory, including the curve constants, which render the equation homogeneous with respect to the weights. To find these weights consider the mapping $\chi \mapsto t^{\alpha_\chi} \tilde{\chi}$ acting on all elements in the curve equation. Define the weights as the constants α_i that render the new equation homogeneous with respect to t. The weights of x, y will then be determined up to a constant by $n\alpha_y = s\alpha_x$. To keep with convention we let $\alpha_x = -n$ and $\alpha_y = -s$ so that they are the largest negative integers satisfying this condition. The weights of the curve constants can then be chosen to make the remainder of the equation homogeneous.

Definition 2.1. *For the cyclic (4,5)-case we have*

	x	y	λ_4	λ_3	λ_2	λ_1	λ_0
Weight	-4	-5	-4	-8	-12	-16	-20

while in the general (4,5)-case the weights of the curve constants are given by their subscripts. We refer to these as the Sato weights.

As we proceed through the paper we can use the approach of this mapping to conclude that other elements in our theory must have definite weight. All the equations presented here are homogeneous with respect to these weights.

Next we construct the standard basis of holomorphic differentials upon C:

$$
\mathbf{d}u = (du_1, \ldots, du_6), \qquad du_i(x, y) = \frac{g_i(x, y)}{4y^3} \, dx,
$$

where

$$
\begin{aligned}
& g_1(x, y) = 1, && g_2(x, y) = x, && g_3(x, y) = y, \\
& g_4(x, y) = x^2, && g_5(x, y) = xy, && g_6(x, y) = y^2.
\end{aligned} \tag{9}
$$

Denote points in $\mathbb{C}^6$ by $\boldsymbol{u}$, for example, and their coordinates by $(u_1, u_2, \ldots, u_6)$. We know from the general theory that any point $\boldsymbol{u} \in \mathbb{C}^6$ can be expressed as

$$\boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5, u_6) = \sum_{i=1}^{6} \int_{\infty}^{P_i} \mathbf{d}\boldsymbol{u},$$

where the P_i are six variable points upon C. Let Λ denote the lattice generated by the integrals of the basis of holomorphic differentials along any closed paths in C. The manifold $\mathbb{C}^6/\Lambda$ is then the Jacobian variety of C, denoted by J. Let κ be the quotient map of modulo Λ over $\mathbb{C}$:

$$\kappa : \mathbb{C}^6 \to \mathbb{C}^6/\Lambda = J.$$

Therefore $\Lambda = \kappa^{-1}((0, \ldots, 0))$. For $k = 1, 2, \ldots$ define $\mathfrak{A}$, the *Abel map* from the kth symmetric product $\mathrm{Sym}^k(C)$ to J:

$$\mathfrak{A} : \mathrm{Sym}^k(C) \to J$$
$$(P_1, \ldots, P_k) \mapsto \left(\int_{\infty}^{P_1} \mathbf{d}\boldsymbol{u} + \cdots + \int_{\infty}^{P_k} \mathbf{d}\boldsymbol{u} \right) \quad (\mathrm{mod}\ \Lambda), \tag{10}$$

where the P_i are points upon C. Denote the image of the kth Abel map by $W^{[k]}$ and let

$$[-1](u_1, \ldots, u_6) = (-u_1, \ldots, -u_6).$$

Define the *kth standard theta subset* (often referred to as the *kth strata*) by

$$\Theta^{[k]} = W^{[k]} \cup [-1]W^{[k]}.$$

When $k = 1$ the Abel map gives an embedding of the curve C upon which we define ξ as the local parameter at the origin, $\mathfrak{A}_1(\infty)$:

$$\xi = x^{-\frac{1}{n}} = x^{-\frac{1}{4}}. \tag{11}$$

We can then express the basis (9) with ξ and integrate to give

$$
\begin{array}{lll}
u_1 = -\frac{1}{11}\xi^{11} + O(\xi^{15}) & u_3 = -\frac{1}{6}\xi^6 + O(\xi^{10}) & u_5 = -\frac{1}{2}\xi^2 + O(\xi^6) \\
u_2 = -\frac{1}{7}\xi^7 + O(\xi^{11}) & u_4 = -\frac{1}{3}\xi^3 + O(\xi^7) & u_6 = -\xi + O(\xi^5).
\end{array}
\tag{12}
$$

The higher-order terms will contain the curve parameters $\boldsymbol{\lambda} = \{\lambda_0, \ldots, \lambda_4\}$.

Similarly to [7, 10] we could rewrite this using u_6 as the local parameter.

Note that such calculations can be performed similarly for any (n, s)-curve, and that since each element of $\mathbf{d}\boldsymbol{u}$ is homogeneous in Sato weight we can conclude that the u_i have definite Sato weight. Since the weight of ξ must be $+1$ from (11), we can define the weights of $\boldsymbol{u}$ uniquely as below.

Definition 2.2. *In the (4,5)-case we assign the following weights to $\boldsymbol{u}$:*

	u_1	u_2	u_3	u_4	u_5	u_6
Weight	$+11$	$+7$	$+6$	$+3$	$+2$	$+1$.

Remark 2.3. The weights of the variables coincide with the order of their zero at ∞. They can also be calculated using the Weierstrass gap sequence, where the weights of $u_1, \ldots, u_6$ are the gap numbers and the weights of x and y are the negative of the first two non-gap numbers.

Definition 2.4. *Let (x, y) and (z, w) be two variable points upon C. Then the two-form $\Omega((x, y), (z, w))$ on $C \times C$ is a fundamental differential of the second kind if*

(1) It is symmetric: $\quad \Omega((x, y), (z, w)) = \Omega((z, w), (x, y))$.

(2) The only pole of second order is along the diagonal of $C \times C$ (where $x = z$).

(3) It can be expanded in a power series as

$$\Omega((x, y), (z, w)) = \left(\frac{1}{(\xi - \xi')^2} + O(1) \right) d\xi \, d\xi' \quad (as \ (x, y) \to (z, w)),$$

where ξ and ξ' are the local coordinates of (x, y) and (z, w).

We will construct Klein's explicit realization of this in proposition 2.6 below.

First introduce $\mathbf{d}r$, the basis of meromorphic differentials which have their only pole at ∞. These are determined modulo the space spanned by the $\mathbf{d}u$ and can be expressed as

$$\mathbf{d}r = (dr_1, \ldots, dr_6), \qquad \text{where} \quad dr_j(x, y) = \frac{h_j(x, y)}{4y^3} \, dx. \tag{13}$$

An explicit basis is constructed later in order to satisfy proposition 2.6.

Definition 2.5. *Define the following meromorphic function on $C \times C$ as*

$$\Sigma((x, y), (z, w)) = \frac{1}{4y^3(x - z)} \cdot \sum_{k=1}^{4} y^{4-k} \left[\frac{f(z, w)}{w^{4-k+1}} \right]_w,$$

where $[\ \ ]_w$ *means that we remove any terms which have negative powers with respect to w.*

Proposition 2.6. *The fundamental differential of the second kind can be expressed as*

$$\Omega((x, y), (z, w)) = R((x, y), (z, w)) \, dx \, dz,$$

where

$$R((x, y), (z, w)) = \frac{\partial}{\partial z} \Sigma((x, y), (z, w)) + \sum_{j=1}^{6} \frac{du_j(x, y)}{dx} \cdot \frac{dr_j(z, w)}{dz}.$$

The polynomials $h_j(x, y)$ need to be chosen so that Ω is symmetric. This will lead to a realization of Ω in the form

$$\Omega((x, y), (z, w)) = \frac{F((x, y), (z, w)) \, dx \, dz}{(x - z)^2 f_y(x, y) f_w(x, y)}. \tag{14}$$

Proof. The essential part of the proof is the same as in the lower genus cases (see [8] for example). In this case, we explicitly determine the basis of meromorphic differentials (13) to be given with

$h_1 = -y^2(8x^2\lambda_4 + 11x^3 + 5x\lambda_3 + 2\lambda_2), \qquad h_2 = -y^2(\lambda_3 + 4x\lambda_4 + 7x^2),$

$h_3 = -2xy(\lambda_3 + 3x^2 + 2x\lambda_4), \qquad h_4 = -3xy^2, \qquad h_5 = -2x^2y, \qquad h_6 = -x^3.$

The polynomial F in the realization (14) is found to be

$$\begin{aligned}
F((x, y), (z, w)) = \ & 4y^3 w^3 + (3xz^4 + z^3\lambda_3 + z^3x^2 + 2\lambda_2 z^2 + 3x\lambda_3 z^2 \\
& + 4z^3 x\lambda_4 + 4\lambda_0 + \lambda_1 x + 2\lambda_2 xz + 3\lambda_1 z)y^2 + (2\lambda_1 z + 4\lambda_2 xz + 4\lambda_0 \\
& + 2\lambda_1 x + 4x^2\lambda_4 z^2 + 2\lambda_3 x^2 z + 2x^3 z^2 + 2z^3 x^2 + 2x\lambda_3 z^2)wy \\
& + (\lambda_3 x^3 + 4\lambda_0 + 3\lambda_1 x + 2\lambda_2 x^2 + \lambda_1 z + x^3 z^2 + 3x^4 z + 2\lambda_2 xz \\
& + 3\lambda_3 x^2 z + 4\lambda_4 x^3 z)w^2.
\end{aligned} \tag{15}$$

$\square$

3. Defining the σ-function

In this section we describe the multivariate σ-function associated with C, from which all Abelian functions associated with C can be defined. This can be regarded as a generalization of the Weierstrass σ-function with the main difference that there are now $g = 6$ variables:

$$\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6).$$

First we choose a basis of cycles (closed paths) upon the surface defined by C. We denote them

$$\alpha_i, \beta_j, \qquad 1 \leqslant i, j \leqslant 6,$$

and ensure they have intersection numbers

$$\alpha_i \cdot \alpha_j = 0, \qquad \beta_i \cdot \beta_j = 0, \qquad \alpha_i \cdot \beta_j = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

This allows us to define the following period matrices:

$$\omega' = \left(\oint_{\alpha_k} \mathrm{d}u_\ell \right)_{k,\ell=1,\ldots,6} \qquad \omega'' = \left(\oint_{\beta_k} \mathrm{d}u_\ell \right)_{k,\ell=1,\ldots,6}$$

$$\eta' = \left(\oint_{\alpha_k} \mathrm{d}r_\ell \right)_{k,\ell=1,\ldots,6} \qquad \eta'' = \left(\oint_{\beta_k} \mathrm{d}r_\ell \right)_{k,\ell=1,\ldots,6}.$$

We combine these into

$$M = \begin{pmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{pmatrix},$$

which we know from classical results to satisfy

$$M \begin{pmatrix} & -I_6 \\ I_6 & \end{pmatrix}^T M = 2\pi \mathrm{i} \begin{pmatrix} & -I_6 \\ I_6 & \end{pmatrix}. \tag{16}$$

This is the *generalized Legendre equation* (see [4], p 11). We also have that $(\omega')^{-1}\omega''$ is symmetric with

$$\mathrm{Im}((\omega')^{-1}\omega'') \quad \text{positive definite.} \tag{17}$$

We now define the multivariate σ-function associated with C. This can be constructed using the multivariate θ-function (see for example [15]).

Definition 3.1. *The Kleinian σ-function associated with C is*

$$\sigma(\mathbf{u}) = \sigma(u; M) = c \exp\left(-\frac{1}{2} \boldsymbol{u}\eta'(\omega')^{-1}\boldsymbol{u}^T \right) \times \theta[\delta]((\omega')^{-1}\boldsymbol{u}^T | (\omega')^{-1}\omega'')$$

$$= c \exp\left(-\frac{1}{2} \boldsymbol{u}\eta'(\omega')^{-1}\boldsymbol{u}^T \right) \times \sum_{m \in \mathbb{Z}^6} \exp\left\{ 2\pi \mathrm{i}\left[\frac{1}{2}(m + \delta')^T (\omega')^{-1}\omega''(m + \delta') \right. \right.$$

$$\left. \left. + (m + \delta')^T ((\omega')^{-1}\boldsymbol{u}^T + \delta'') \right] \right\},$$

where c is a constant dependent upon the curve parameters $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and fixed later (see remark 4). The matrix $\delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix}$ is the θ-function characteristic which gives the Riemann constant for C with respect to the base point ∞ and the period matrix $[\omega', \omega'']$ (see [4], pp 23–4).

In this paper we give some of the most important properties of $\sigma(\boldsymbol{u})$. For a more detailed study of the construction and properties of the multivariate σ-function we refer the reader to [4].

Lemma 3.2. *Given $\boldsymbol{u} \in \mathbb{C}^6$, denote by $\boldsymbol{u}'$ and $\boldsymbol{u}''$ the unique elements in $\mathbb{R}^6$ such that*

$$\boldsymbol{u} = \boldsymbol{u}'\omega' + \boldsymbol{u}''\omega''.$$

Let ℓ represent a point on the period lattice

$$\ell = \ell'\omega' + \ell''\omega'' \in \Lambda.$$

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^6$ and $\ell \in \Lambda$, define $L(\boldsymbol{u}, \boldsymbol{v})$ and $\chi(\ell)$ as follows:

$$L(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}^T(\eta'\boldsymbol{v}' + \eta''\boldsymbol{v}''),$$
$$\chi(\ell) = \exp[\pi i(2(\ell'^{\,T}\delta'' - \ell''^{\,T}\delta') + \ell'^{\,T}\ell'')].$$

Then for all $\boldsymbol{u} \in \mathbb{C}^6, \ell \in \Lambda$ the function $\sigma(\boldsymbol{u})$ has the quasi-periodicity property

$$\sigma(\boldsymbol{u} + \ell) = \chi(\ell)\exp\left[L\left(\boldsymbol{u} + \frac{\ell}{2}, \ell\right)\right] \cdot \sigma(\boldsymbol{u}). \tag{18}$$

Also for $\gamma \in Sp(12, \mathbb{Z})$ we have

$$\sigma(\boldsymbol{u}; \gamma M) = \sigma(\boldsymbol{u}; M). \tag{19}$$

Proof. The quasi-periodicity property given in (18) is a classical result, first discussed in [2], that was fundamental to the original definition of the multivariate σ-function. Equation (19) is easily seen from the definition of $\sigma(\boldsymbol{u})$, since γ corresponds to the choice of basis cycles $\{\alpha_j, \beta_j\}_{j=1}^6$ which are used to define M. $\square$

4. Classes of Abelian functions

Definition 4.1. *Let $\mathfrak{M}(\boldsymbol{u})$ be a meromorphic function of $\boldsymbol{u} \in \mathbb{C}^6$. Then $\mathfrak{M}$ is an Abelian function associated with C if*

$$\mathfrak{M}(\boldsymbol{u} + \omega'\boldsymbol{n}^T + \omega''\boldsymbol{m}^T) = \mathfrak{M}(\boldsymbol{u})$$

for all integer vectors $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}$, wherever $\mathfrak{M}$ is defined.

We now define a set of fundamental Abelian functions on J.

Definition 4.2. *Define the 2-index Kleinian $\wp$-functions as*

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j}\log\sigma(\boldsymbol{u}), \qquad i \leqslant j \in \{1, \ldots, 6\}.$$

A short calculation shows these functions to have poles of order 2 when $\sigma(\boldsymbol{u}) = 0$ and no other singularities. We can check (using lemma 3.2) that

$$\wp_{ij}(\boldsymbol{u} + \ell) = \wp_{ij}(\boldsymbol{u}), \qquad \forall \quad \ell \in \Lambda.$$

Hence we can conclude these functions to be Abelian. Similar analysis will show their derivatives to be Abelian also.

Definition 4.3. *For $n \geqslant 2$ define n-index Kleinian $\wp$-functions as*

$$\wp_{i_1, i_2, \ldots, i_n}(\boldsymbol{u}) = -\frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}}\cdots\frac{\partial}{\partial u_{i_n}}\log\sigma(\boldsymbol{u}), \qquad i_1 \leqslant \cdots \leqslant i_n \in \{1, \ldots, 6\}.$$

Remark 4.4.

  (i) Compare with (1) and the elliptic case to see that we are defining a generalization of the Weierstrass $\wp$-function and its derivatives.
 (ii) This notation is compatible with the elliptic case, where we would now denote the Weierstrass $\wp$-function as $\wp_{11}(\boldsymbol{u})$ and its first derivative $\wp'$ by $\wp_{111}(\boldsymbol{u})$.
(iii) The order of the indices is irrelevant. For simplicity we always use ascending numerical order.
(iv) We are usually only referring to one vector of variables $\boldsymbol{u}$. In these cases, for simplicity, we write $\wp_{ij}$ instead of $\wp_{ij}(\boldsymbol{u})$.

We find in section 7 that the $\wp$-functions are not sufficient to construct a basis of the simplest Abelian functions. Hence we also define a generalization of Baker's Q-functions, which we need to extend further than in the lower genus cases.

Definition 4.5. *Define the operator Δ_i as below. This is now known as Hirota's bilinear operator, although it was used much earlier by Baker in [3]:*

$$\Delta_i = \frac{\partial}{\partial u_i} - \frac{\partial}{\partial v_i}.$$

It is then simple to check that an alternative, equivalent definition of the 2-index Kleinian $\wp$-functions is given by

$$\wp_{ij}(\boldsymbol{u}) = -\frac{1}{2\sigma(\boldsymbol{u})^2}\Delta_i\Delta_j\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\bigg|_{v=u} \qquad i \leqslant j \in \{1,\ldots,6\}.$$

We extend this to define n-index Q-functions, for n even, by

$$Q_{i_1,i_2,\ldots,i_n}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2}\Delta_{i_1}\Delta_{i_2}\ldots\Delta_{i_n}\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\bigg|_{v=u}$$

where $i_1 \leqslant \cdots \leqslant i_n \in \{1,\ldots,6\}$.

We can show as above that these functions are also Abelian.

Remark 4.6.

  (i) The subscripts of the $\wp$-functions denote differentiation

$$\frac{\partial}{\partial u_{i_{n+1}}}\wp_{i_1,i_2,\ldots,i_n} = \wp_{i_1,i_2,\ldots,i_n,i_{n+1}},$$

but this is not the case for the Q-functions. Here the indices refer to which Hirota operators were used.
 (ii) If we had applied the definition for n odd then it would have returned zero.
(iii) Note that both the $\wp$-functions and the Q-functions have poles when $\sigma(\boldsymbol{u}) = 0$ and no other singularities. The n-index $\wp$-functions have poles of order n, while the n-index Q-function all have poles of order 2.

The 4-index Q-functions were first used by Baker, and in [8] it was shown that they could be expressed using the Kleinian $\wp$-functions as

$$Q_{ijk\ell} = \wp_{ijk\ell} - 2\wp_{ij}\wp_{k\ell} - 2\wp_{ik}\wp_{j\ell} - 2\wp_{i\ell}\wp_{jk}. \tag{20}$$

Proposition 4.7. *The 6-index Q-functions can be written as*

$$
\begin{aligned}
Q_{ijklmn} = {} & \wp_{ijklmn} - 2[(\wp_{ij}\wp_{klmn} + \wp_{ik}\wp_{jlmn} + \wp_{il}\wp_{jkmn} + \wp_{im}\wp_{jkln} \\
& + \wp_{in}\wp_{jklm}) + (\wp_{jk}\wp_{ilmn} + \wp_{jl}\wp_{ikmn} + \wp_{jm}\wp_{ikln} + \wp_{jn}\wp_{iklm}) \\
& + (\wp_{kl}\wp_{ijmn} + \wp_{km}\wp_{ijln} + \wp_{kn}\wp_{ijlm}) + (\wp_{lm}\wp_{ijkn} + \wp_{ln}\wp_{ijkm}) \\
& + \wp_{mn}\wp_{ijkl}] + 4[(\wp_{ij}\wp_{kl}\wp_{mn} + \wp_{ij}\wp_{km}\wp_{ln} + \wp_{ij}\wp_{kn}\wp_{lm}) \\
& + (\wp_{ik}\wp_{jl}\wp_{mn} + \wp_{ik}\wp_{jm}\wp_{ln} + \wp_{ik}\wp_{jn}\wp_{lm}) + (\wp_{il}\wp_{jk}\wp_{mn} \\
& + \wp_{il}\wp_{jm}\wp_{kn} + \wp_{il}\wp_{jn}\wp_{km}) + (\wp_{im}\wp_{jk}\wp_{ln} + \wp_{im}\wp_{jl}\wp_{kn} \\
& + \wp_{im}\wp_{jn}\wp_{kl}) + (\wp_{in}\wp_{jk}\wp_{lm} + \wp_{in}\wp_{jl}\wp_{km} + \wp_{in}\wp_{jm}\wp_{kl})].
\end{aligned}
\tag{21}
$$

Proof. Apply definitions 4.2 and 4.5 to reduce the equation to a sum of σ-derivatives. We find that they all cancel (Maple is useful here). The structure of the sum was prompted by considering the result for the 4-index Q-functions. $\square$

Clearly (21) will specialize to give a set of simpler formulae such as

$$
Q_{nnnnnn} = \wp_{nnnnnn} - 30\wp_{nn}\wp_{nnnn} + 60\wp_{nn}^3.
$$

5. Expanding the Kleinian formula

This section is based upon the following theorem (originally by Klein). It is given for a general curve as theorem 3.4 in [7]. From this theorem we are able to solve the Jacobi inversion problem, as well as generate relations between the $\wp$-functions.

Theorem 5.1. *Let $\{P_1, \ldots, P_6\} \in C^6$ be an arbitrary set of distinct points on C, and (z, w) any point of this set. Then for an arbitrary point (x, y) and base point ∞ on C we have*

$$
\sum_{i,j=1}^{6} \wp_{ij} \left(\int_{\infty}^{(x,y)} \mathbf{d}u - \sum_{k=1}^{6} \int_{\infty}^{P_k} \mathbf{d}u \right) g_i(x, y) g_j(z, w) = \frac{F\big((x, y), (z, w)\big)}{(x - z)^2}.
\tag{22}
$$

Here g_i is the numerator of $\mathrm{d}u_i$ as given in (9), and F is the symmetric function appearing in (15) as the numerator of the fundamental differential of the second kind.

We use our explicit calculation of the differentials to construct (22). We expand this as one of the P_k tends to infinity to obtain a series expansion in terms of the local parameter ξ, given earlier in (11). It follows that each coefficient with respect to ξ must be zero for any $\mathbf{u} \in J$ and some (z, w) on C. This gives us a potentially infinite sequence of equations starting with the five given in appendix A. The first 14 have been calculated explicitly (using Maple) and can be found online at [16].

Manipulating these equations. We follow the approach of the trigonal papers and manipulate these equations using Maple. We first take the resultant of pairs of equations (eliminating the variable w by choice) to give a new set of equations dependent on z and the $\wp$-functions. We introduce the notation $\mathrm{Res}(a, b)$ to represent the resultant of equations (a) and (b).

These new equations are considerably longer than those obtained in the lower genus cases. We need to combine them to give a polynomial of degree $g - 1 = 5$ in z. Such a polynomial would have only five solutions but must be satisfied for all $\mathbf{u}$ (which has six variables). Hence all the coefficients must be zero, giving us a set of relations between the $\wp$-functions. We have the extra complication (compared to the trigonal cases) that none of the new equations

has degree in z equal to g. Therefore, in this case, at least two rounds of elimination between the equations will be required.

We find that Res(A.1, A.2) has degree 7 in z so we rearrange it to give an equation for z^7. Then since Res(A.1, A.3) and Res(A.1) and (A.5) have degree 8, we can repeatedly substitute for z^7 in both until we are left with two equations of degree 6 in z. Since these are very long we do not print them here; however, they can be found online at [16] where we have labelled them (T1) and (T2).

We next rearrange (T1) to give an equation for z^6 and repeatedly substitute for z^7 and z^6 in the remaining equations until they are of degree 5 in z. The coefficients of such equations must be zero, giving us relations between the $\wp$-functions. The smallest such relation has 3695 terms, with the others rising in size considerably. Unlike the trigonal cases these cannot be easily separated to give expressions for individual $\wp$-functions. However, these are implemented in the construction of the σ-function expansion (see section 6).

Jacobi inversion problem. Recall that the Jacobi inversion problem is, given a point $\boldsymbol{u} \in J$, to find the preimage of this point under the Abel map (10).

Theorem 5.2. *Suppose we are given* $\{u_1, \ldots, u_6\} = \boldsymbol{u} \in J$. *Then we could solve the Jacobi inversion problem explicitly using the equations derived from (A.1)–(A.5).*

Proof. Consider either equation (T1) or (T2) defined in the discussion above. This is a polynomial constructed from $\wp$-functions and the variable z. This equation has degree 6 in z so denote by $(z_1, \ldots, z_6)$ the six zeros of the polynomial.

Next, rearrange (A.1) to give an equation for w^2. Substitute this into (A.2) and multiply all terms by $\wp_{66}$ to give the following equation of degree 1 with respect to w:

$$
\begin{aligned}
0 = w\big(&z\wp_{66}\wp_{55} - 2z^2\wp_{66} - z\wp_{66}\wp_{566} + \wp_{666}z\wp_{56} + \wp_{36}\wp_{666} + \wp_{66}\wp_{35} \\
&- z\wp_{56}^2 - \wp_{36}\wp_{56} - \wp_{66}\wp_{366}\big) + z^2\wp_{66}\wp_{45} - z^2\wp_{66}\wp_{466} + \wp_{56}z^3 \\
&- z\wp_{66}\wp_{266} - \wp_{56}z\wp_{26} + z\wp_{66}\wp_{25} + \wp_{15}\wp_{66} - \wp_{666}z^3 + \wp_{666}\wp_{46}z^2 \\
&+ \wp_{666}z\wp_{26} - \wp_{166}\wp_{66} + \wp_{666}\wp_{16} - \wp_{56}\wp_{46}z^2 - \wp_{56}\wp_{16}.
\end{aligned} \tag{23}
$$

We could substitute each z_i into (23) in turn and solve to find the corresponding w_i. We can therefore identify the set of points $\{(z_1, w_1), \ldots, (z_6, w_6)\}$ on the curve C which are the Abel preimage of $\boldsymbol{u}$. $\square$

6. Deriving the properties of $\sigma(\boldsymbol{u})$

In this section, we derive some of the properties for $\sigma(\boldsymbol{u})$ and use them to construct the Taylor series expansion.

Lemma 6.1. *The function* $\sigma(\boldsymbol{u})$ *has zeroes of order 1 when* $\boldsymbol{u} \in \Theta^{[5]}$. *Further,* $\sigma(\boldsymbol{u}) \neq 0$ *for all other* $\boldsymbol{u}$.

Proof. This is a classical result which always holds for $\Theta^{[g-1]}$ (see [2]). It can be concluded from results on Riemann's θ-function, from which $\sigma(\boldsymbol{u})$ can be defined.

The first part can also be concluded explicitly from the results of the previous section. In theorem 5.2 we discussed how the six roots of the polynomial (T2) gave us the Abel preimage of $\boldsymbol{u} \in J$. Now suppose that $\boldsymbol{u}$ is approaching $\Theta^{[5]}$, implying one of these roots is approaching infinity. We explicitly calculate the denominator of (T2) to be $\sigma(\boldsymbol{u})^{16}$, using definition 4.3. Therefore, we can conclude that when $\boldsymbol{u}$ descends to $\Theta^{[5]}$ we must have $\sigma(\boldsymbol{u}) = 0$. $\square$

Consider $u \in \Theta^{[5]}$ which by definition we can express using points P_k on C as

$$u = \int_\infty^{P_1} \mathbf{d}u + \cdots + \int_\infty^{P_5} \mathbf{d}u.$$

Use equations (12) to express u with five local parameters:

$$0 = u_1 + \frac{1}{11}\xi_1^{11} + \cdots + \frac{1}{11}\xi_5^{11} + O(\xi_1^{15}) + \cdots + O(\xi_5^{15}),$$

$$\vdots$$

$$0 = u_6 + \xi_1 + \cdots + \xi_5 + O(\xi_1^5) + \cdots + O(\xi_5^5). \tag{24}$$

Now consider the case when $\lambda = 0$. In this case equations (12) simplify to

$$u_1 = -\frac{1}{11}\xi^{11}, \quad u_2 = -\frac{1}{7}\xi^7, \quad u_3 = -\frac{1}{6}\xi^6, \quad u_4 = -\frac{1}{3}\xi^3, \quad u_5 = -\frac{1}{2}\xi^2, \quad u_6 = -\xi,$$

and hence the higher-order terms in equations (24) all reduce to zero. We can now take the multivariate resultant of these six finite polynomials, eliminating the parameters $\xi_1, \ldots, \xi_5$. From the theory of resultants we are left with a polynomial, unique up to multiplication by a non-vanishing holomorphic function, that must be zero for $u \in \Theta^{[5]}$. By lemma 6.1 we can conclude this polynomial to be a multiple of $\sigma(u)$.

In fact, this is just a specific case of the following stronger result for the σ-function.

Lemma 6.2. *Define the canonical limit of the σ-function as the value of $\sigma(u)$ in the case when all the curve constants are zero. In this case the series expansion of $\sigma(u)$ about $u = (0,0,0,0,0,0)$ is given by a constant K multiplied by the Schur–Weierstrass polynomial generated by (n, s).*

Proof. The result was first stated in [17], with an alternative proof now available in [18]. □

For the $(4, 5)$-case we have the following Schur–Weierstrass polynomial:

$$
\begin{aligned}
SW_{4,5} = {}& \tfrac{1}{83\,82\,528}u_6^{15} + \tfrac{1}{336}u_6^8 u_5^2 u_4 - \tfrac{1}{12}u_6^4 u_1 - \tfrac{1}{126}u_6^7 u_3 u_5 - \tfrac{1}{6}u_4 u_3 u_5 u_6^4 \\
& - \tfrac{1}{72}u_4^3 u_6^6 - \tfrac{1}{33\,264}u_6^{11}u_5^2 + \tfrac{1}{27}u_5^6 u_6^3 + \tfrac{2}{3}u_4 u_5^3 u_3 - 2u_4^2 u_6 u_3 u_5 - u_2^2 u_6 \\
& - \tfrac{2}{9}u_5^3 u_3 u_6^3 - u_4 u_3^2 + \tfrac{1}{12}u_4^4 u_6^3 - \tfrac{1}{3\,024}u_6^9 u_4^2 - \tfrac{1}{756}u_6^7 u_5^4 + \tfrac{1}{1\,008}u_6^8 u_2 \\
& + \tfrac{1}{3}u_5^4 u_2 + \tfrac{1}{3}u_6^3 u_3^2 - \tfrac{1}{9}u_4 u_5^6 + \tfrac{1}{399\,168}u_6^{12}u_4 + u_4 u_6 u_5^2 u_2 + \tfrac{1}{4}u_4^5 \\
& + 2u_5 u_3 u_2 + \tfrac{1}{6}u_5^2 u_6^4 u_2 + \tfrac{1}{12}u_6^5 u_2 u_4 - \tfrac{1}{2}u_4^2 u_6^2 u_2 + \tfrac{1}{2}u_4^3 u_6^2 u_5^2 \\
& - \tfrac{1}{3}u_4^2 u_6 u_5^4 - \tfrac{1}{36}u_5^4 u_4 u_6^4 + u_4 u_6 u_1 - u_5^2 u_1, \tag{25}
\end{aligned}
$$

which, as expected, is a factor in the polynomial obtained from the resultant calculation described above. Note that calculating $SW_{4,5}$ as the Schur–Weierstrass polynomial is, computationally, far easier than using a multivariate resultant method.

In the discussion above we could have truncated equations (12) at successively higher weights of λ, instead of just setting $\lambda = 0$. In each case we would use resultants to generate a polynomial that is a multiple of the expansion of $\sigma(u)$, truncated at that weight. Since this was generated using polynomials of homogeneous weight we know that the expansion of $\sigma(u)$ must also have definite weight.

Remark 6.3. We will fix the constant c in definition (3.1) to be the value that makes $K = 1$ in lemma (6.2). Some other authors working in this area would instead set

$$c = \left(\frac{\pi^6}{\det(w')}\right)^{\frac{1}{2}} \cdot \frac{1}{D^{\frac{1}{8}}},$$

where D is the discriminant of the curve C. In general, these two choices of c are not equivalent. However the constant c will cancel in the definitions of all the Abelian functions, and hence any relation between such functions is independent of c.

Lemma 6.4. *The function $\sigma(u)$ associated with the (4,5)-curve is odd with respect to $u \mapsto [-1]u$.*

Proof. In theorem 3(iii) of [18] the author shows that for any (n, s)-curve

$$\sigma(-u) = (-1)^{\frac{1}{24}(n^2-1)(s^2-1)}\sigma(u).$$

Hence in the (4,5)-case the function $\sigma(u)$ must be odd.   □

We now have enough information to derive a Taylor series expansion for $\sigma(u)$, similar to that of the elliptic case in (4).

Theorem 6.5. *The function $\sigma(u)$ associated with (7) has the following expansion:*

$$\sigma(\mathbf{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6) = C_{15}(u) + C_{19}(u) + \cdots + C_{15+4n}(u) + \cdots,$$

where each C_k is a finite, odd polynomial composed of products of monomials in $u = (u_1, \ldots, u_6)$ of total weight $+k$ multiplied by monomials in $\lambda = (\lambda_4, \ldots, \lambda_0)$ of total weight $15 - k$.

Proof. The theoretical part of the proof follows [8, 9]. By theorem 3(i) in [18], we know the expansion will be a sum of monomials in u and λ with rational coefficients, and by lemma 6.4 we conclude that the expansion must be odd. We also know that $\sigma(u)$ has definite weight, and by lemma 6.2 we can conclude this to be the same weight as the Schur–Weierstrass polynomial. From equation (25) we see this weight is +15.

The rationale of the construction is that although the expansion is homogeneous of weight +15, it will contain both u (with positive weight) and λ (with negative weight). We hence split up the infinite expansion into finite polynomials whose terms share common weight ratios.

The first polynomial will be the terms with the lowest weight in u. These must be the terms that do not vary with λ. The indices then increase by four since the weights of λ decrease by four (see definition 2.1).   □

By lemma 6.2 we have $C_{15} = SW_{4,5}$ as given by (25). Using the computer algebra package Maple we calculate the other polynomials successively as follows:

 (i) Select the terms that could appear in C_k. These are a finite number of monomials formed by entries of u and λ with the appropriate weight ratio.
 (ii) Construct $\hat{\sigma}(u)$ as the sum of C_k derived thus far. Then add to this each of the possible terms multiplied by an independent, unidentified constant.
 (iii) Determine the constants by ensuring $\hat{\sigma}(u)$ satisfies known properties of the σ-function:
  - For the first few C_k this is mainly ensuring lemma 6.1 is satisfied (as in the trigonal calculations).
  - For the latter C_k the coefficients are instead determined by ensuring a variety of the equations from lemma 7.4 are satisfied.
  - In addition, those polynomials up to C_{39} require we ensure $\sigma(u)$ satisfies some of the relations between $\wp$-functions obtained from the expansion of the Kleinian formula in section 5.

The second method is the most computationally efficient (due to the pole cancellations), while the third method is extremely difficult. The equations in lemma 7.4 are derived in tandem with the σ-function expansion, and so cannot be used for the first few C_k.

The expansion has been calculated up to and including C_{59}. Appendix B contains C_{19} and C_{23} with the rest of the expansion online at [16]. These latter polynomials are extremely large and represent a significant amount of computation. Many of the calculations were run in parallel on a cluster of machines using the Distributed Maple package (see [11]). This expansion is sufficient for any explicit calculations. However, it would be ideal to find a recursive construction of the expansion generalizing the elliptic case (see for example [19]).

7. Relations between the Abelian functions

In the previous section, we showed that $\sigma(\boldsymbol{u})$ has definite Sato weight, and hence so do the Abelian functions defined from it. We can conclude from definition 4.3 that

$$\text{wt}(\wp_{i_1, i_2, \ldots, i_n}) = -[\text{wt}(u_{i_1}) + \text{wt}(u_{i_2}) + \cdots + \text{wt}(u_{i_n})]. \tag{26}$$

Then use equations (20) and (21) respectively to conclude

$$\text{wt}(Q_{ijkl}) = \text{wt}(\wp_{ijkl}) \qquad \text{and} \qquad \text{wt}(Q_{ijklmn}) = \text{wt}(\wp_{ijklmn}). \tag{27}$$

We now introduce the following definition to classify the Abelian functions associated with C by their pole structure.

Definition 7.1. *Define*

$$\Gamma(J, \mathcal{O}(m\Theta^{[k]}))$$

as the vector space of Abelian functions defined upon J which have poles of order at most m, occurring only on the kth standard theta subset, $\Theta^{[k]}$.

Recall that the Abelian functions we define all have poles occurring only when $\sigma(\boldsymbol{u}) = 0$, which by lemma 6.1, is when $\boldsymbol{u} \in \Theta^{[5]}$. Therefore, using remark 4.6 (iii) we conclude that the n-index $\wp$-functions belong to $\Gamma(J, \mathcal{O}(n\Theta^{[5]}))$, while the n-index Q-functions all belong to $(J, \mathcal{O}(2\Theta^{[5]}))$.

Lemma 7.2. *Suppose we have a basis for the vector space $\Gamma(J, \mathcal{O}(m\Theta^{[k]}))$. Then an element of the space that is not contained in the basis can be expressed as a linear combination of the basis entries, with coefficients polynomial in $\boldsymbol{\lambda} = \{\lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_0\}$.*

Proof. The significance of the lemma is that we need not consider the coefficients to be rational functions of $\boldsymbol{\lambda}$, as may be expected. We can modify the argument from theorem 9.1 in [8] to prove this. Let X be an element of the vector space that is not in the basis. Then

$$X = \sum_j A_j(\boldsymbol{\lambda}) Y_j \equiv \sum_j \frac{P_j(\boldsymbol{\lambda})}{Q_j(\boldsymbol{\lambda})} Y_j, \tag{28}$$

where the Y_j are elements of the basis, and the A_j are rational functions of $\boldsymbol{\lambda}$. Since the polynomials P_j, Q_j belong to a polynomial ring we suppose that the A_j have been expressed in reduced fractional form. We will suppose for a contradiction that at least one of the A_j is not polynomial.

Define B as the least common multiple of $\{Q_j(\boldsymbol{\lambda})\}$. There will be specific values of $\boldsymbol{\lambda}$ that set $B = 0$ while leaving at least one $P_j(\boldsymbol{\lambda})$ non-zero (see, for example, chapter 1 of [20]). Multiply both sides of equation (28) by B and take $\boldsymbol{\lambda}$ to be one of these special values. In this case the equation we obtain would invalidate the linear independence of the basis. Therefore we conclude that all the A_j must be polynomial in $\boldsymbol{\lambda}$.　　　□

Theorem 4. *A basis for* $\Gamma(J, \mathcal{O}(2\Theta^{[5]}))$ *is given by*

$$
\begin{array}{ccccccccc}
 & \mathbb{C}1 & \oplus & \mathbb{C}\wp_{11} & \oplus & \mathbb{C}\wp_{12} & \oplus & \mathbb{C}\wp_{13} & \oplus & \mathbb{C}\wp_{14} \\
\oplus & \mathbb{C}\wp_{15} & \oplus & \mathbb{C}\wp_{16} & \oplus & \mathbb{C}\wp_{22} & \oplus & \mathbb{C}\wp_{23} & \oplus & \mathbb{C}\wp_{24} \\
\oplus & \mathbb{C}\wp_{25} & \oplus & \mathbb{C}\wp_{26} & \oplus & \mathbb{C}\wp_{33} & \oplus & \mathbb{C}\wp_{34} & \oplus & \mathbb{C}\wp_{35} \\
\oplus & \mathbb{C}\wp_{36} & \oplus & \mathbb{C}\wp_{44} & \oplus & \mathbb{C}\wp_{45} & \oplus & \mathbb{C}\wp_{46} & \oplus & \mathbb{C}\wp_{55} \\
\oplus & \mathbb{C}\wp_{56} & \oplus & \mathbb{C}\wp_{66} & \oplus & \mathbb{C}Q_{5566} & \oplus & \mathbb{C}Q_{4556} & \oplus & \mathbb{C}Q_{4555} \\
\oplus & \mathbb{C}Q_{4455} & \oplus & \mathbb{C}Q_{3566} & \oplus & \mathbb{C}Q_{3556} & \oplus & \mathbb{C}Q_{2566} & \oplus & \mathbb{C}Q_{2556} \\
\oplus & \mathbb{C}Q_{3456} & \oplus & \mathbb{C}Q_{2456} & \oplus & \mathbb{C}Q_{3366} & \oplus & \mathbb{C}Q_{3445} & \oplus & \mathbb{C}Q_{2366} \\
\oplus & \mathbb{C}Q_{2445} & \oplus & \mathbb{C}Q_{1466} & \oplus & \mathbb{C}Q_{1556} & \oplus & \mathbb{C}Q_{2266} & \oplus & \mathbb{C}Q_{2356} \\
\oplus & \mathbb{C}Q_{2256} & \oplus & \mathbb{C}Q_{2346} & \oplus & \mathbb{C}Q_{1455} & \oplus & \mathbb{C}Q_{2345} & \oplus & \mathbb{C}Q_{3344} \\
\oplus & \mathbb{C}Q_{2245} & \oplus & \mathbb{C}Q_{2344} & \oplus & \mathbb{C}Q_{1266} & \oplus & \mathbb{C}Q_{1356} & \oplus & \mathbb{C}Q_{1444} \\
\oplus & \mathbb{C}Q_{1346} & \oplus & \mathbb{C}Q_{2236} & \oplus & \mathbb{C}Q_{2335} & \oplus & \mathbb{C}Q_{1246} & \oplus & \mathbb{C}Q_{1255} \\
\oplus & \mathbb{C}Q_{1245} & \oplus & \mathbb{C}Q_{1166} & \oplus & \mathbb{C}Q_{1244} & \oplus & \mathbb{C}Q_{1156} & \oplus & \mathbb{C}Q_{1146} \\
\oplus & \mathbb{C}Q_{1155} & \oplus & \mathbb{C}Q_{1145} & \oplus & \mathbb{C}Q_{1144} & \oplus & \mathbb{C}Q_{114466}.
\end{array}
\tag{29}
$$

Proof. The dimension of the space is $2^g = 2^6 = 64$ by the Riemann–Roch theorem for Abelian varieties. It was shown above that all the selected elements do, in fact, belong to the space. All that remains is to prove their linear independence, which can be done explicitly using Maple. □

The actual construction of the basis is as follows. We start by including all 21 of the $\wp_{ij}$ in the basis, since they are all linearly independent. Then, to decide which Q_{ijkl} to include, we systematically consider decreasing weights in turn, starting at -4 since this is the highest weight of any Q-function. At each stage we derive equations to express the Q-functions at that weight using the following method (implemented with Maple):

 (i) We form a sum of basis entries, each multiplied by an undetermined coefficient. We include those basis entries at this weight, along with elements in the basis of a higher weight (already determined) combined with appropriate λ-monomials that balance the weight. Note from lemma 7.2 that we need not consider basis entries multiplied by rational functions in the $\boldsymbol{\lambda}$.
 (ii) We also include in this sum the Q_{ijkl} which are at this weight.
(iii) Substitute the Abelian functions for their definitions as σ-derivatives.
(iv) Substitute $\sigma(\boldsymbol{u})$ for the expansion, truncated at the appropriate point.
 (v) Take the numerator of the resulting expression and separate into monomials in $\boldsymbol{u}$ and $\boldsymbol{\lambda}$, with coefficients in the unidentified coefficients.
(vi) Set all the coefficients to zero and solve the resulting system of equations.

At weights which have more than one Q-function we often find that one or more must be added to the basis so that the others can be expressed.

We form these equations at successively lower weights constructing the basis as we proceed. As the weight decreases we require more of the expansion, which is why these were calculated in tandem. Also, as the weight decreases the possible number of terms increases, and the computations take more time and memory. Upon completing this process we have 63 basis elements.

We find the final element by considering the 6-index Q-functions. Repeating the process we found that one of the functions at weight -30 is required to express the others.

Sets of differential equations satisfied by the Abelian functions. We now present a number of differential equations between the Abelian functions. The number in brackets on the left indicates the weight of the equation.

Lemma 7.4. *Those 4-index Q-functions not in the basis can be expressed as a linear combination of the basis elements:*

$$(-4) \quad Q_{6666} = -3\wp_{55} + 4\wp_{46},$$

$$(-5) \quad Q_{5666} = -2\wp_{45},$$

$$(-6) \quad Q_{4666} = 6\lambda_4\wp_{66} - 2\wp_{44} - \tfrac{3}{2}Q_{5566},$$

$$(-7) \quad Q_{4566} = 2\lambda_4\wp_{56} + 2\wp_{36},$$

$$(-7) \quad Q_{5556} = 4\lambda_4\wp_{56} + 4\wp_{36},$$

$$\vdots$$

A longer list is given in appendix C, while the full set is available online at [16].

The same statement is also true for all the 6-index Q-functions, except Q_{114466} which is in the basis. Explicit relations have been calculated down to weight -30. The first few are given below with all available relations online at [16]:

$$(-6) \quad Q_{666666} = 40\wp_{44} + 15Q_{5566} - 24\wp_{66}\lambda_4,$$

$$(-7) \quad Q_{566666} = 20\wp_{36} - 4\wp_{56}\lambda_4,$$

$$(-8) \quad Q_{556666} = 24\wp_{26} - 12\wp_{35} - 2Q_{4556},$$

$$(-8) \quad Q_{466666} = -20\wp_{35} + 5Q_{4556} + 16\wp_{46}\lambda_4 - 20\wp_{55}\lambda_4 - 8\lambda_3.$$

Proof. By lemma 7.2 it is clear that such relations must exist. The explicit PDEs were calculated in the construction of the basis, as discussed at the start of this section. □

Corollary 7.5. *There are a set of PDEs that express 4-index $\wp$-functions using Abelian functions of order at most 2. The full set can be found online at [16]:*

$$(-4) \quad \wp_{6666} = 6\wp_{66}^2 - 3\wp_{55} + 4\wp_{46}, \tag{30a}$$

$$(-5) \quad \wp_{5666} = 6\wp_{56}\wp_{66} - 2\wp_{45}, \tag{30b}$$

$$(-6) \quad \wp_{4666} = 6\wp_{46}\wp_{66} + 6\lambda_4\wp_{66} - 2\wp_{44} - \tfrac{3}{2}\wp_{5566} + 3\wp_{66}\wp_{55} + 6\wp_{56}^2,$$

$$(-7) \quad \wp_{4566} = 2\wp_{45}\wp_{66} + 4\wp_{46}\wp_{56} + 2\lambda_4\wp_{56} + 2\wp_{36},$$

$$(-7) \quad \wp_{5556} = 6\wp_{55}\wp_{56} + 4\lambda_4\wp_{56} + 4\wp_{36}.$$

Proof. Apply (20) to the first set of relations in lemma 7.4. □

The set of equations in corollary 7.5 is of particular interest because it gives a generalization of (2) from the elliptic case. A similar generalization for (3) would be a set of equations that express the 3-index $\wp$-functions using Abelian function of order at most 3. So far the following

relations have been derived (see [16] for the latest list):

$(-\mathbf{6})\quad \wp_{666}^2 = 4\wp_{66}^3 - 7\wp_{56}^2 + 4\wp_{46}\wp_{66} - 8\wp_{55}\wp_{66} - 4\wp_{66}\lambda_4 + 4\wp_{44} + 2\wp_{5566},$

$(-\mathbf{7})\quad \wp_{566}\wp_{666} = 4\wp_{66}^2\wp_{56} + 2\wp_{46}\wp_{56} - \wp_{55}\wp_{56} - 2\wp_{45}\wp_{66} + 2\wp_{36},$

$(-\mathbf{8})\quad \wp_{556}\wp_{666} = -4\wp_{26} - 2\wp_{35} - 4\wp_{55}\lambda_4 - 4\lambda_3 + 2\wp_{4556} - 6\wp_{45}\wp_{56}$
$\qquad\qquad\qquad\quad - 2\wp_{46}\wp_{55} + \wp_{5566}\wp_{66} - 2\wp_{56}^2\wp_{66} - 2\wp_{55}^2,$

$(-\mathbf{8})\quad \wp_{566}^2 = 4\wp_{56}^2\wp_{66} + 4\wp_{46}\wp_{55} + \wp_{55}^2 + 4\wp_{55}\lambda_4 + 4\wp_{45}\wp_{56} + 8\wp_{26}$
$\qquad\qquad\qquad\quad + 4\lambda_3 - 2\wp_{4556},$

$(-\mathbf{8})\quad \wp_{466}\wp_{666} = 4\wp_{56}^2\wp_{66} + 4\wp_{46}\wp_{66}^2 + 2\wp_{55}\wp_{66}^2 + 4\wp_{66}^2\lambda_4 + 2\wp_{46}^2$
$\qquad\qquad\qquad\quad - 2\wp_{55}\lambda_4 - 3\wp_{45}\wp_{56} - 2\wp_{44}\wp_{66} - \wp_{5566}\wp_{66} - 2\wp_{26} - 2\wp_{35}$
$\qquad\qquad\qquad\quad - 4\wp_{46}\wp_{55} - 2\lambda_3 + \wp_{4556},$

$(-\mathbf{9})\quad \wp_{556}\wp_{566} = -2\wp_{56}^3 - 2\wp_{45}\wp_{55} - \tfrac{4}{3}\wp_{45}\lambda_4 + \wp_{5566}\wp_{56} - \tfrac{4}{3}\wp_{34} + \tfrac{1}{3}\wp_{4555},$

$(-\mathbf{9})\quad \wp_{555}\wp_{666} = -3\wp_{5566}\wp_{56} - 4\wp_{44}\wp_{56} + 8\wp_{56}\wp_{66}\lambda_4 + 12\wp_{56}\wp_{55}\wp_{66}$
$\qquad\qquad\qquad\quad + 10\wp_{56}^3 + \tfrac{8}{3}\wp_{45}\lambda_4 - \tfrac{4}{3}\wp_{34} - \tfrac{2}{3}\wp_{4555} + 4\wp_{45}\wp_{46} + 8\wp_{36}\wp_{66},$

$(-\mathbf{9})\quad \wp_{466}\wp_{566} = \tfrac{2}{3}\wp_{34} - 2\wp_{25} - 2\wp_{45}\wp_{46} + \tfrac{1}{3}\wp_{4555} + 2\wp_{56}\wp_{55}\wp_{66} - \wp_{45}\wp_{55}$
$\qquad\qquad\qquad\quad + 4\wp_{56}\wp_{46}\wp_{66} + 4\wp_{56}^3 - \tfrac{4}{3}\wp_{45}\lambda_4 - 2\wp_{44}\wp_{56} + 4\wp_{56}\wp_{66}\lambda_4 - \wp_{5566}\wp_{56},$

$(-\mathbf{9})\quad \wp_{456}\wp_{666} = -\wp_{56}\wp_{55}\wp_{66} + 2\wp_{45}\wp_{66}^2 + 2\wp_{56}\wp_{46}\wp_{66} + 2\wp_{36}\wp_{66} - 2\wp_{56}^3$
$\qquad\qquad\qquad\quad + 2\wp_{44}\wp_{56} + \tfrac{4}{3}\wp_{45}\lambda_4 + \tfrac{1}{2}\wp_{5566}\wp_{56} - \tfrac{2}{3}\wp_{34} - \tfrac{1}{3}\wp_{4555} + 2\wp_{45}\wp_{46}.$

Proposition 7.6. *There are a set of relations that are bilinear in the 2-index and 3-index $\wp$-functions. (See [16] for full list.) There is no analogue in the elliptic case, although similar relations have been derived in the hyperelliptic and trigonal cases:*

$(-\mathbf{6})\quad 0 = -\wp_{555} + 2\wp_{456} + 2\wp_{566}\wp_{66} - 2\wp_{56}\wp_{666},$

$(-\mathbf{7})\quad 0 = -2\wp_{446} + 2\wp_{455} - 2\wp_{466}\wp_{66} + 2\wp_{666}\lambda_4 + 2\wp_{46}\wp_{666}$
$\qquad\qquad\quad - 2\wp_{556}\wp_{66} + \wp_{55}\wp_{666} + \wp_{566}\wp_{56},$

$(-\mathbf{8})\quad 0 = 2\wp_{46}\wp_{566} - 2\wp_{56}\wp_{466} + \wp_{555}\wp_{66} - 2\wp_{55}\wp_{566} + \wp_{556}\wp_{56} - 2\wp_{366},$

$(-\mathbf{8})\quad 0 = 2\wp_{456}\wp_{66} - \wp_{445} + \wp_{56}\wp_{466} - \wp_{366} - \wp_{566}\lambda_4$
$\qquad\qquad\quad - \wp_{45}\wp_{666} - 2\wp_{46}\wp_{566},$

$(-\mathbf{9})\quad 0 = -2\wp_{455}\wp_{66} + 4\wp_{266} + 2\wp_{45}\wp_{566} + 2\wp_{466}\wp_{55} - 2\wp_{46}\wp_{556}$
$\qquad\qquad\quad - \wp_{555}\wp_{56} + \wp_{556}\wp_{55} - 2\wp_{356}.$

$\hfill(31)$

Proof. These can be calculated by cross differentiating suitable pairs of equations from corollary 7.5. For example, (30) expresses $\wp_{6666}(\boldsymbol{u})$ while (30) expresses $\wp_{5666}(\boldsymbol{u})$. If we substitute for these equations into

$$\frac{\partial}{\partial u_5}\wp_{6666}(\boldsymbol{u}) - \frac{\partial}{\partial u_6}\wp_{5666}(\boldsymbol{u}) = 0,$$

then we find (31).　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　$\square$

A topic of future work in this area would be the construction of relations between the $\wp$-functions in covariant form, as was recently achieved in [21] for the hyperelliptic case.

8. Solution to the KP-equation

We now demonstrate how such Abelian functions can give a solution to the KP-equation. Differentiate (30) twice with respect to u_6 to obtain

$$\wp_{666666} = 12 \frac{\partial}{\partial u_6} (\wp_{66} \wp_{666}) - 3\wp_{5566} + 4\wp_{4666}.$$

Make the substitutions $u_6 = x, u_5 = y, u_4 = t$ and $W(x, y, t) = \wp_{66}(u)$, and rearrange to give the following parametrized form of the KP-equation:

$$[W_{xxx} - 12WW_x - 4W_t]_x + 3W_{yy} = 0.$$

In fact, this is just a special case of the following general result for Abelian functions associated with algebraic curves.

Theorem 8.1. *Let E be an (n, s)-curve with genus g as given by (6). Define the multivariate σ-function associated with E as normal. Define the Abelian functions from $\sigma(u)$ as in section 4 (with the indices now running to g instead of 6). Finally, define the function $W(u) = \wp_{gg}(u)$, which we denote $W(x, y, t)$ after applying the substitutions $u_g = x, u_{g-1} = y, u_{g-2} = t$.*

Then, if $n \geqslant 4$, the function $W(x, y, t)$ will satisfy the following parametrized version of the KP-equation:

$$(W_{xxx} - 12WW_x - bW_t)_x - aW_{yy} = 0, \qquad (32)$$

for some constants a, b.

Proof. Recall definitions 2.1 and 2.2 which gave the Sato weights for C. These can be calculated for the general curve E similarly as

$$\begin{aligned} \mathrm{wt}(x) &= -n, & \mathrm{wt}(y) &= -s, \\ \mathrm{wt}(u_g) &= \omega_1, & \mathrm{wt}(u_{g-1}) &= \omega_2, & \ldots \quad \mathrm{wt}(u_1) &= \omega_g, \\ \mathrm{wt}(\lambda_0) &= -ns, & \mathrm{wt}(\lambda_1) &= -n(s-1), & \ldots \quad \mathrm{wt}(\lambda_{s-1}) &= -n. \end{aligned}$$

Here n, s are the integers generating the curve and $\{\omega_1, \ldots, \omega_g\}$ is the Weierstrass gap sequence for n, s. These are the natural numbers not representable in the form $an + bs$ where $a, b \in \mathbb{N}$. (See [17], section 1 for more details.)

Since $s > n > 4$ we know that $\{1, 2, 3\}$ cannot be represented in this form. Therefore we have

$$\mathrm{wt}(u_g) = +1, \qquad \mathrm{wt}(u_{g-1}) = +2, \qquad \mathrm{wt}(u_{g-2}) = +3.$$

By (26) this implies the $\wp$-functions will have weights

$$\mathrm{wt}(\wp_{g,g}) = -2, \qquad \mathrm{wt}(\wp_{g,g-1}) = -3, \qquad \mathrm{wt}(\wp_{g-1,g-1}) = -4, \qquad \mathrm{wt}(\wp_{g-2,g}) = -4,$$

with all the other 2-index $\wp$-functions having a lower weight. Next consider Q_{gggg}, which will have weight -4. This will belong to $\Gamma(J, \mathcal{O}(2\Theta^{[g-1]}))$, the space of Abelian functions defined upon the Jacobian of E which have poles of at most order 2 on $\Theta^{[g-1]}$. By lemma 7.2 we can express Q_{gggg} as

$$Q_{gggg} = a\wp_{g-1,g-1} + b\wp_{g-2,g}, \qquad (a, b \text{ constants}),$$

since these are the only Abelian functions of weight -4, and there is no function of lower weight that could be combined with a λ-monomial. We use remark 4.6 to substitute for Q and then differentiate twice with respect to u_g to give

$$\wp_{gggggg} = 12 \frac{\partial}{\partial u_g} (\wp_{gg} \wp_{ggg}) + a\wp_{g-1,g-1,g,g} + b\wp_{g-2,g,g,g}.$$

Then make the substitutions suggested in the theorem to obtain (32). $\qquad \square$

Further research into the applications of these results is currently being conducted.

9. Two-term addition formula

Theorem 9.1. *The functions associated with (7) satisfy the following two-term addition formula:*

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = f(u,v) - f(v,u),$$

where $f(u,v)$ is a polynomial of Abelian functions, given in appendix D.

Proof. We seek to express the following ratio of σ-functions (labelled LHS) using a sum of Abelian functions:

$$\text{LHS}(u,v) = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}. \tag{33}$$

First recall that $\sigma(u)$ is an odd function with respect to the change of variables $u \mapsto [-1]u$. We use this to consider the effect of $(u,v) \mapsto (v,u)$ on LHS:

$$\text{LHS}(v,u) = -\frac{\sigma(u+v)\sigma([-1](u-v))}{\sigma(u)^2\sigma(v)^2} = -\text{LHS}(u,v).$$

So LHS is antisymmetric or odd with respect to $(u,v) \mapsto (v,u)$.

Next, recall that $\sigma(u)$ has zeros of order 1 along $\Theta^{[5]}$ and no zeros elsewhere. This implies that LHS has poles of order 2 along

$$(\Theta^{[5]} \times J) \cup (\Theta^{[5]} \times J)$$

but nowhere else. Together this implies that we can express LHS as

$$\text{LHS} = \sum_j A_j(X_j(u)Y_j(v) - X_j(v)Y_j(u)), \tag{34}$$

where the X_j and Y_j are functions chosen from the basis in theorem 4, and the A_j are constant coefficients. A modification of lemma 7.2 will show that the A_j must be polynomial functions of λ.

Finally, we use the fact that σ has weight $+15$ to determine that the weight of LHS is -30. Hence we need only consider those terms in (34) that give the correct overall weight.

We use Maple to construct (34) with the A_j undetermined. This contained 1348 terms (647 undetermined coefficients since it is antisymmetric). The coefficients can be determined using the σ-function expansion. $\qquad\square$

We believe this to be the first of a family of similar addition formula related to the invariance expressed in (8). There has been much work conducted into these addition formula for the trigonal cases (see [8] for example). In [22], we see that this has inspired new results in the lower genus cases.

Acknowledgments

Appendix A. Expansion of the Kleinian formula

We consider (22) from theorem 5.1. We expand this as one of the P_k tends to infinity to obtain a series expansion in terms of the local parameter ξ. It follows that each coefficient with respect to ξ must be zero, giving us an infinite sequence of equations starting with those below:

$$0 = -z^3 + \wp_{46}z^2 + (\wp_{56}w + \wp_{26})z + \wp_{66}w^2 + \wp_{36}w + \wp_{16,} \tag{A.1}$$

$$0 = (\wp_{45} - \wp_{466} - 2w)z^2 + ((\wp_{55} - \wp_{566})w + \wp_{25} - \wp_{266})z \\ + (\wp_{56} - \wp_{666})w^2 + (\wp_{35} - \wp_{366})w + \wp_{15} - \wp_{166,} \tag{A.2}$$

$$0 = \left(\wp_{44} - \tfrac{3}{2}\wp_{456} + \tfrac{1}{2}\wp_{4666}\right)z^2 + \left(-3w^2 + \left(\wp_{45} - \tfrac{3}{2}\wp_{556} + \tfrac{1}{2}\wp_{5666}\right)w \\ - \tfrac{3}{2}\wp_{256} + \wp_{24} + \tfrac{1}{2}\wp_{2666}\right)z + \left(\tfrac{1}{2}\wp_{6666} - \tfrac{3}{2}\wp_{566} + \wp_{46}\right)w^2 \\ + \left(\tfrac{1}{2}\wp_{3666} + \wp_{34} - \tfrac{3}{2}\wp_{356}\right)w + \wp_{14} + \tfrac{1}{2}\wp_{1666} - \tfrac{3}{2}\wp_{156,} \tag{A.3}$$

$$0 = \left(\wp_{4566} - \tfrac{4}{3}\wp_{446} - \tfrac{1}{6}\wp_{46666} - \tfrac{1}{2}\wp_{455}\right)z^2 + \left(\left(\wp_{5566} - \tfrac{4}{3}\wp_{456} - \tfrac{1}{6}\wp_{56666}\right.\right. \\ \left.- \tfrac{1}{2}\wp_{555}\right)w - \tfrac{4}{3}\wp_{246} - \tfrac{1}{2}\wp_{255} + \wp_{2566} - \tfrac{1}{6}\wp_{26666}\right)z - 4w^3 + \left(\wp_{5666}\right. \\ \left.- \tfrac{1}{2}\wp_{556} - \tfrac{1}{6}\wp_{66666} - \tfrac{4}{3}\wp_{466}\right)w^2 + \left(-\tfrac{1}{6}\wp_{36666} + \wp_{3566} - \tfrac{4}{3}\wp_{346}\right. \\ \left.- \tfrac{1}{2}\wp_{355}\right)w - \tfrac{1}{6}\wp_{16666} + \wp_{1566} - \tfrac{4}{3}\wp_{146} - \tfrac{1}{2}\wp_{155,} \tag{A.4}$$

$$0 = -3z^4 - \left(2\wp_{46} + \tfrac{9}{2}\lambda_4\right)z^3 + \left(\tfrac{5}{8}\wp_{4556} - \tfrac{5}{6}\wp_{445} - \tfrac{5}{12}\wp_{45666} - 2\wp_{56}w - 2\wp_{26}\right. \\ \left.- 3\lambda_3 + \tfrac{5}{6}\wp_{4466} + \tfrac{1}{2}\wp_{46}\lambda_4 + \tfrac{1}{24}\wp_{466666}\right)z^2 + \left(-2\wp_{66}w^2 + \left(\tfrac{1}{24}\wp_{566666}\right.\right. \\ \left.- \tfrac{5}{6}\wp_{455} + \tfrac{5}{6}\wp_{4566} - \tfrac{5}{12}\wp_{55666} + \tfrac{1}{2}\wp_{56}\lambda_4 + \tfrac{5}{8}\wp_{5556} - 2\wp_{36}\right)w - \tfrac{5}{12}\wp_{25666} \\ \left.- \tfrac{5}{6}\wp_{245} - 2\lambda_2 + \tfrac{5}{6}\wp_{2466} + \tfrac{5}{8}\wp_{2556} + \tfrac{1}{24}\wp_{266666} - 2\wp_{16} + \tfrac{1}{2}\wp_{26}\lambda_4\right)z \\ + \left(\tfrac{1}{2}\wp_{66}\lambda_4 - \tfrac{5}{12}\wp_{56666} + \tfrac{1}{24}\wp_{666666} - \tfrac{5}{6}\wp_{456} + \tfrac{5}{6}\wp_{4666} + \tfrac{5}{8}\wp_{5566}\right)w^2 \\ + \left(\tfrac{1}{24}\wp_{366666} + \tfrac{5}{6}\wp_{3466} + \tfrac{5}{8}\wp_{3556} + \tfrac{1}{2}\wp_{36}\lambda_4 - \tfrac{5}{12}\wp_{35666} - \tfrac{5}{6}\wp_{345}\right)w \\ + \tfrac{5}{8}\wp_{1556} - \lambda_1 - \tfrac{5}{12}\wp_{15666} + \tfrac{1}{24}\wp_{166666} + \tfrac{5}{6}\wp_{1466} - \tfrac{5}{6}\wp_{145} + \tfrac{1}{2}\wp_{16}\lambda_4. \tag{A.5}$$

Appendix B. The σ-function expansion

We defined the σ-function expansion as the infinite sum of finite polynomials:

$$\sigma(\mathbf{u}) = C_{15} + C_{19} + C_{23} + C_{27} + C_{31} + C_{35} + \cdots.$$

We know that C_{15} is equal to the Schur–Weierstrass polynomial as given in (25). The other polynomials were calculated, in turn, using the method described in section 6. The next polynomials, C_{19} and C_{23}, are given below, while the rest of the expansion can be found at [16]:

$$C_{19} = \lambda_4 \cdot \left[\tfrac{1}{13970880}u_6{}^{16}u_4 + \tfrac{2}{135}u_6{}^3u_5{}^8 - u_6{}^2u_5u_4{}^3u_3 - u_6{}^2u_4u_2{}^2 + \tfrac{4}{45}u_5{}^6u_2 \right. \\ - \tfrac{2}{45}u_6{}^3u_5{}^5u_3 - \tfrac{1}{5}u_6u_5{}^6u_4{}^2 + \tfrac{1}{90}u_6{}^4u_5{}^6u_4 - \tfrac{1}{30}u_6{}^5u_4u_1 + \tfrac{5}{3024}u_6{}^9u_5{}^2u_4{}^2 \\ + \tfrac{1}{10}u_5{}^2u_4{}^5 + \tfrac{1}{30}u_6{}^5u_2{}^2 + \tfrac{1}{20956320}u_6{}^{15}u_5{}^2 - \tfrac{1}{3}u_5{}^4u_1 - \tfrac{2}{45}u_5{}^8u_4 \\ \left. - \tfrac{1}{1890}u_6{}^7u_5{}^6 - \tfrac{1}{120}u_6{}^7u_4{}^4 + \tfrac{1}{665280}u_6{}^{13}u_4{}^2 - \tfrac{1}{83160}u_6{}^{11}u_5{}^4 \right.$$

$$- \frac{1}{630} u_6{}^7 u_3{}^2 - \frac{1}{5040} u_6{}^{10} u_4{}^3 + \frac{1}{3024} u_6{}^9 u_4 u_2 + \frac{1}{2} u_6{}^2 u_4{}^2 u_1 + u_6{}^2 u_5{}^2 u_4{}^2 u_2$$

$$- \frac{1}{280} u_6{}^8 u_5 u_4 u_3 + \frac{2}{15} u_6{}^5 u_5{}^2 u_4 u_2 - \frac{2}{9} u_6{}^4 u_5{}^3 u_4 u_3 - \frac{2}{3} u_6 u_5{}^3 u_4{}^2 u_3$$

$$- \frac{1}{10} u_6{}^5 u_5 u_4{}^2 u_3 + \frac{1}{3} u_6 u_5{}^4 u_4 u_2 - u_6 u_4{}^2 u_3{}^2 + \frac{1}{20} u_6{}^4 u_4{}^5 + \frac{1}{2520} u_6{}^8 u_5{}^4 u_4$$

$$+ \frac{1}{30} u_6{}^6 u_4{}^2 u_2 + \frac{1}{6} u_6{}^4 u_4 u_3{}^2 + \frac{4}{3} u_5{}^3 u_3 u_2 - \frac{4}{945} u_6{}^7 u_5{}^3 u_3 + \frac{1}{18} u_6{}^4 u_5{}^4 u_2$$

$$- \frac{1}{180} u_6{}^6 u_5{}^2 u_4{}^3 - \frac{17}{997920} u_6{}^{12} u_5{}^2 u_4 + \frac{1}{630} u_6{}^8 u_5{}^2 u_2 + \frac{3}{20} u_6 u_4{}^6$$

$$+ \frac{2}{15} u_5{}^5 u_4 u_3 + \frac{1}{2} u_4{}^4 u_2 + \frac{1}{997920} u_6{}^{12} u_2 + \frac{1}{5040} u_6{}^8 u_1$$

$$- \frac{1}{6} u_6{}^3 u_4{}^3 u_2 - \frac{1}{83160} u_6{}^{11} u_5 u_3 - \frac{1}{60} u_6{}^5 u_5{}^4 u_4{}^2 + \frac{1}{3} u_6{}^3 u_5{}^2 u_4{}^4 \Big],$$

$$C_{23}(\boldsymbol{u}) = \lambda_3 \cdot \Big[u_5{}^2 u_3{}^2 u_2 - \frac{1}{360} u_6{}^6 u_4{}^2 u_1 + \frac{1}{6} u_5{}^4 u_4 u_3{}^2 - \frac{1}{18} u_6{}^3 u_5{}^4 u_3{}^2$$

$$- \frac{1}{252} u_6{}^7 u_5{}^2 u_3{}^2 + \frac{1}{72} u_6{}^4 u_5{}^4 u_1 + \frac{1}{6} u_6 u_5{}^4 u_2{}^2 + \frac{1}{12} u_6{}^5 u_5{}^2 u_2{}^2$$

$$+ \frac{1}{12} u_6{}^4 u_3{}^2 u_2 + \frac{1}{2016} u_6{}^8 u_5{}^2 u_1 \frac{5}{144} u_6{}^4 u_4{}^4 u_2 - \frac{29}{17962560} u_6{}^{12} u_5{}^4 u_4$$

$$+ \frac{1}{11176 7040} u_6{}^{16} u_5{}^2 u_4 + \frac{17}{11975040} u_6{}^{13} u_4 u_2 - \frac{17}{315} u_5{}^7 u_4 u_3 - \frac{1}{30} u_6{}^5 u_5 u_4 u_3 u_2$$

$$+ \frac{1}{18} u_6{}^3 u_4{}^3 u_1 + \frac{1}{5} u_5{}^5 u_3 u_2 - \frac{1}{74844} u_6{}^{12} u_5{}^2 u_2 - \frac{1}{6} u_6{}^3 u_4{}^2 u_2{}^2$$

$$+ \frac{1}{630} u_6 u_5{}^8 u_4{}^2 + \frac{1}{7560} u_6{}^4 u_5{}^8 u_4 - \frac{1}{45360} u_6{}^8 u_5{}^6 u_4 - \frac{1}{60} u_6 u_5{}^2 u_4{}^6$$

$$+ \frac{1}{240} u_6{}^4 u_5{}^2 u_4{}^5 + \frac{1}{216} u_6{}^3 u_5{}^4 u_4{}^4 + \frac{1}{1680} u_6{}^7 u_5{}^2 u_4{}^4 + \frac{2}{15} u_5 u_4{}^5 u_3$$

$$+ \frac{1}{6480} u_6{}^6 u_5{}^4 u_4{}^3 + \frac{1}{36288} u_6{}^{10} u_5{}^2 u_4{}^3 + \frac{1}{299376} u_6{}^{11} u_5{}^3 u_3 - \frac{1}{2} u_6{}^2 u_2{}^3$$

$$- \frac{1}{540} u_6{}^2 u_5{}^6 u_4{}^3 + \frac{1}{25147584} u_6{}^{15} u_5 u_3 - \frac{1}{1260} u_6{}^7 u_5{}^5 u_3 + \frac{1}{6} u_4{}^3 u_2{}^2$$

$$+ \frac{1}{335301120} u_6{}^{17} u_4{}^2 - \frac{1}{630} u_5{}^8 u_2 - \frac{1}{1890} u_5{}^{10} u_4 + \frac{1}{60} u_6{}^5 u_4{}^2 u_3{}^2$$

$$- \frac{11}{10080} u_6{}^8 u_4 u_3{}^2 + \frac{7}{60} u_6 u_4{}^5 u_2 - \frac{37}{181440} u_6{}^{10} u_4{}^2 u_2 - \frac{13}{1890} u_6{}^7 u_4{}^3 u_2$$

$$+ \frac{17}{945} u_6{}^3 u_5{}^7 u_3 + \frac{1}{12} u_5{}^2 u_4{}^4 u_2 + \frac{2}{135} u_6{}^4 u_5{}^6 u_2 - \frac{1}{3024} u_6{}^8 u_5{}^4 u_2$$

$$+ \frac{1}{158760} u_6{}^7 u_5{}^8 - \frac{1}{6} u_6 u_5{}^4 u_4 u_1 + \frac{1}{1995840} u_6{}^{13} u_5{}^2 u_4{}^2 + \frac{1}{120} u_6{}^2 u_4{}^7 + \cdots$$

$$\cdots + \frac{5}{18144} u_6{}^9 u_2{}^2 + \frac{1}{114672 98304} u_6{}^{19} u_5{}^2 + \frac{1}{2395008} u_6{}^{12} u_1 + \frac{2}{3} u_6 u_5{}^3 u_4 u_3 u_2$$

$$+ \frac{1}{5670} u_6{}^3 u_5{}^{10} - \frac{1}{18} u_6{}^4 u_5{}^3 u_3 u_2 + \frac{1}{9} u_6{}^2 u_5{}^3 u_4{}^3 u_3 - \frac{1}{5} u_6 u_5{}^5 u_4{}^2 u_3$$

$$- \frac{43}{5987520} u_6{}^{12} u_5 u_4 u_3 + \frac{113}{90720} u_6{}^9 u_5{}^2 u_4 u_2 - \frac{1}{22934596 6080} u_6{}^{20} u_4$$

$$- \frac{1}{90} u_6{}^5 u_5{}^3 u_4{}^2 u_3 + \frac{1}{540} u_6{}^5 u_5{}^6 u_4{}^2 + \frac{1}{2} u_6{}^2 u_5{}^2 u_4 u_2{}^2 + \frac{1}{17010} u_6{}^9 u_5{}^4 u_4{}^2$$

$$- \frac{1}{8064} u_6{}^8 u_4{}^5 - \frac{7}{360} u_5{}^4 u_4{}^5 - \frac{1}{3991680} u_6{}^{11} u_4{}^4 - \frac{1}{75442752} u_6{}^{15} u_5{}^4$$

$$- \frac{1}{199584} u_6{}^{11} u_3{}^2 + \frac{1}{28740096} u_6{}^{16} u_2 + \frac{1}{720} u_6{}^5 u_4{}^6 - \frac{1}{16765 0560} u_6{}^{14} u_4{}^3$$

$$- \frac{1}{12} u_6{}^4 u_5{}^2 u_4 u_3{}^2 - \frac{1}{1008} u_6{}^8 u_5 u_3 u_2 + \frac{1}{2} u_6{}^2 u_5{}^2 u_4{}^2 u_1 + \frac{11}{90720} u_6{}^9 u_4 u_1$$

$$+ \frac{11}{360} u_6{}^6 u_4 u_2{}^2 - u_6{}^2 u_5 u_4{}^2 u_3 u_2 - u_6 u_4 u_3{}^2 u_2 - u_6 u_5{}^2 u_4{}^2 u_3{}^2$$

$$+ \frac{1}{9} u_6{}^3 u_5 u_4{}^4 u_3 + \frac{11}{15120} u_6{}^8 u_5{}^3 u_4 u_3 + \frac{1}{60} u_6{}^5 u_5{}^2 u_4 u_1 - \frac{7}{90} u_6 u_5{}^6 u_4 u_2$$

$$- \frac{7}{360} u_6{}^5 u_5{}^4 u_4 u_2 - \frac{7}{360} u_6{}^6 u_5{}^2 u_4{}^2 u_2 - \frac{7}{540} u_6 u_5 u_4{}^3 u_3 - \frac{1}{30} u_5{}^6 u_1$$

$$- \frac{1}{60} u_6{}^4 u_5{}^5 u_4 u_3 - \frac{1}{12} u_6{}^2 u_5{}^4 u_4{}^2 u_2 + \frac{1}{45360} u_6{}^9 u_5 u_4{}^2 u_3 + \frac{1}{3} u_4{}^4 u_1$$

$$+ \frac{1}{10595783632896} u_6{}^{23} - \frac{1}{6} u_6{}^2 u_4{}^3 u_3{}^2 + \frac{2}{9} u_6{}^3 u_5{}^2 u_4{}^3 u_2 - \frac{1}{641520} u_6{}^{11} u_5{}^6 \Big]$$

$$+ \lambda_4{}^2 \cdot \Big[- \frac{1}{180} u_6{}^6 u_4{}^2 u_1 - \frac{1}{72} u_6{}^4 u_4{}^4 u_2 - \frac{13}{2041200} u_6{}^{12} u_5{}^4 u_4$$

$$+ \frac{1}{2993760} u_6{}^{13} u_4 u_2 + \frac{4}{315} u_5{}^7 u_4 u_3 + \frac{1}{22680} u_6{}^9 u_4 u_1 + \frac{1}{9} u_6{}^3 u_4{}^3 u_1$$

$$+ \frac{4}{15} u_5{}^5 u_3 u_2 + \frac{1}{467775} u_6{}^{12} u_5{}^2 u_2 - \frac{1}{3} u_6{}^3 u_4{}^2 u_2{}^2 - \frac{17}{315} u_6 u_5{}^8 u_4{}^2$$

$$+ \tfrac{5}{756}u_6{}^4 u_5{}^8 u_4 - \tfrac{1}{14175}u_6{}^8 u_5{}^6 u_4 + \tfrac{1}{15}u_6 u_5{}^2 u_4{}^6 + \tfrac{13}{120}u_6{}^4 u_5{}^2 u_4{}^5$$

$$+ \tfrac{7}{108}u_6{}^3 u_5{}^4 u_4{}^4 - \tfrac{1}{280}u_6{}^7 u_5{}^2 u_4{}^4 - \tfrac{1}{30}u_5 u_4{}^5 u_3 - \tfrac{1}{162}u_6{}^6 u_5{}^4 u_4{}^3$$

$$+ \tfrac{101}{226\,800}u_6{}^{10} u_5{}^2 u_4{}^3 - \tfrac{4}{467\,775}u_6{}^{11} u_5{}^3 u_3 - \tfrac{8}{135}u_6{}^2 u_5{}^6 u_4{}^3$$

$$- \tfrac{4}{4725}u_6{}^7 u_5{}^5 u_3 + \tfrac{1}{3}u_4{}^3 u_2{}^2 + \tfrac{17}{838\,252\,800}u_6{}^{17} u_4{}^2 + \tfrac{1}{105}u_5{}^8 u_2$$

$$+ \tfrac{1}{30}u_6{}^5 u_4{}^2 u_3{}^2 - \tfrac{1}{2520}u_6{}^8 u_4 u_3{}^2 + \tfrac{17}{60}u_6 u_4{}^5 u_2 + \tfrac{1}{16200}u_6{}^{10} u_4{}^2 u_2$$

$$+ \tfrac{23}{3780}u_6{}^7 u_4{}^3 u_2 - \tfrac{4}{945}u_6{}^3 u_5{}^7 u_3 + \tfrac{1}{6}u_5{}^2 u_4{}^4 u_2 + \tfrac{1}{135}u_6{}^4 u_5{}^6 u_2$$

$$- \tfrac{1}{3}u_6{}^2 u_4{}^3 u_3{}^2 - \tfrac{43}{396\,900}u_6{}^7 u_5{}^8 - \tfrac{1}{199\,584}u_6{}^{13} u_5{}^2 u_4{}^2 + \tfrac{1}{24}u_6{}^2 u_4{}^7$$

$$- \tfrac{2}{45}u_5{}^6 u_1 - \tfrac{1}{22680}u_6{}^9 u_2{}^2 - \tfrac{1}{286\,682\,457\,60}u_6{}^{19} u_5{}^2 - \tfrac{1}{149\,688\,00}u_6{}^{12} u_1$$

$$+ \tfrac{41}{14175}u_6{}^3 u_5{}^{10} - \tfrac{4}{9}u_6{}^2 u_5{}^3 u_4{}^3 u_3 - \tfrac{2}{15}u_6 u_5{}^5 u_4{}^2 u_3 - \tfrac{31}{748\,4400}u_6{}^{12} u_5 u_4 u_3$$

$$+ \tfrac{2}{2835}u_6{}^9 u_5{}^2 u_4 u_2 + \tfrac{1}{573\,364\,915\,200}u_6{}^{20} u_4 - \tfrac{4}{45}u_6{}^5 u_5{}^3 u_4{}^2 u_3$$

$$- \tfrac{1}{1350}u_6{}^5 u_5{}^6 u_4{}^2 + \tfrac{29}{680\,40}u_6{}^9 u_5{}^4 u_4{}^2 - \tfrac{247}{100\,800}u_6{}^8 u_4{}^5 + \tfrac{1}{36}u_5{}^4 u_4{}^5$$

$$- \tfrac{593}{997\,9200}u_6{}^{11} u_4{}^4 + \tfrac{1}{673\,596\,00}u_6{}^{15} u_5{}^4 + \tfrac{1}{124\,7400}u_6{}^{11} u_3{}^2 + \tfrac{1}{251\,475\,8400}u_6{}^{16} u_2$$

$$+ \tfrac{13}{900}u_6{}^5 u_4{}^6 + \tfrac{19}{41912640}u_6{}^{14} u_4{}^3 - \tfrac{1}{12}u_4{}^4 u_1$$

$$+ \tfrac{1}{90}u_6{}^6 u_4 u_2{}^2 - \tfrac{5}{18}u_6{}^3 u_5 u_4{}^4 u_3 - \tfrac{2}{945}u_6{}^8 u_5{}^3 u_4 u_3 + \tfrac{2}{45}u_6 u_5{}^6 u_4 u_2$$

$$+ \tfrac{2}{45}u_6{}^5 u_5{}^4 u_4 u_2 + \tfrac{2}{45}u_6{}^6 u_5{}^2 u_4{}^2 u_2 - \tfrac{7}{270}u_6{}^6 u_5 u_4{}^3 u_3 + \tfrac{4}{9}u_6{}^3 u_5{}^2 u_4{}^3 u_2$$

$$- \tfrac{2}{45}u_6{}^4 u_5{}^5 u_4 u_3 + \tfrac{1}{3}u_6{}^2 u_5{}^4 u_4{}^2 u_2 - \tfrac{17}{226\,80}u_6{}^9 u_5 u_4{}^2 u_3$$

$$- \tfrac{1}{561\,330}u_6{}^{11} u_5{}^6 - \tfrac{1}{264\,894\,590\,822\,40}u_6{}^{23} + \tfrac{1}{1890}u_6{}^8 u_5{}^4 u_2$$

$$+ \tfrac{1}{399\,168\,00}u_6{}^{16} u_5{}^2 u_4 - \tfrac{1}{157\,172\,400}u_6{}^{15} u_5 u_3 - \tfrac{41}{4725}u_5{}^{10} u_4 \Big].$$

Appendix C. The 4-index *Q*-functions

This appendix contains a list of PDEs expressing the 4-index Q-functions that were not elements of the basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[5]})\big)$ as a linear combination of basis elements. This appendix contains all the equations down to weight -22. The full set can be accessed at [16]. The PDEs are ordered in decreasing weight (the number in brackets):

(**−4**) $Q_{6666} = -3\wp_{55} + 4\wp_{46},$

(**−5**) $Q_{5666} = -2\wp_{45},$

(**−6**) $Q_{4666} = 6\lambda_4\wp_{66} - 2\wp_{44} - \tfrac{3}{2}Q_{5566},$

(**−7**) $Q_{4566} = 2\lambda_4\wp_{56} + 2\wp_{36},$

(**−7**) $Q_{5556} = 4\lambda_4\wp_{56} + 4\wp_{36},$

(**−8**) $Q_{4466} = 4\lambda_4\wp_{46} + \lambda_4\wp_{55} + \wp_{35} + 6\wp_{26} - Q_{4556} + 4\lambda_3,$

(**−8**) $Q_{5555} = 16\lambda_4\wp_{55} + 4\wp_{35} + 24\wp_{26} - 6Q_{4556} + 16\lambda_3,$

(**−9**) $Q_{4456} = \tfrac{8}{3}\lambda_4\wp_{45} + 2\wp_{25} - \tfrac{4}{3}\wp_{34} - \tfrac{1}{6}Q_{4555},$

(**−9**) $Q_{3666} = 2\lambda_4\wp_{45} - \tfrac{1}{2}Q_{4555},$

(**−10**) $Q_{2666} = -\tfrac{1}{2}Q_{4455} - \tfrac{1}{2}Q_{3566} - \tfrac{1}{2}\lambda_4 Q_{5566} + 4\wp_{66}\lambda_3,$

(**−10**) $Q_{4446} = 6\lambda_4\wp_{44} - 2\wp_{24} - 2\lambda_4^2\wp_{66}$

$$- Q_{4455} + \tfrac{1}{2}\lambda_4 Q_{5566} - \tfrac{5}{2}Q_{3566} + 4\wp_{66}\lambda_3,$$

(-11) $Q_{3466} = 4\wp_{36}\lambda_4 - \frac{1}{2}Q_{3566},$

(-11) $Q_{4445} = 6\wp_{36}\lambda_4 - 2\wp_{56}\lambda_4^2 + 8\wp_{56}\lambda_3 - 3Q_{2566} - \frac{3}{2}Q_{3556},$

(-12) $Q_{2466} = 8\wp_{16} - \wp_{33} + 2\lambda_2 - \frac{1}{2}Q_{2456} - Q_{3456} + 4\lambda_4\wp_{26} + 2\lambda_4\wp_{35},$

(-12) $Q_{3555} = 24\wp_{16} - 8\wp_{33} + 16\lambda_4\wp_{35} - 6Q_{3456},$

(-12) $Q_{4444} = -12\wp_{16} + 9\wp_{33} + 6\lambda_2 + 12Q_{3456} + 12\wp_{55}\lambda_3 + 4\wp_{46}\lambda_4^2$
$\qquad\qquad - 16\wp_{46}\lambda_3 - 3\wp_{55}\lambda_4^2 + 12\lambda_4\wp_{26} - 18\lambda_4\wp_{35} - 6Q_{2556},$

(-13) $Q_{2555} = 2\wp_{15} - 8\wp_{23} + 16\wp_{25}\lambda_4 - 6Q_{2456},$

(-13) $Q_{3446} = 6\wp_{15} - 4\wp_{23} + 4\wp_{25}\lambda_4 + \frac{8}{3}\wp_{34}\lambda_4 + \frac{2}{3}\wp_{45}\lambda_4^2 - \frac{1}{6}Q_{4555}\lambda_4 - 2Q_{2456},$

(-13) $Q_{3455} = -4\wp_{25}\lambda_4 + \frac{8}{3}\wp_{34}\lambda_4 - \frac{4}{3}\wp_{45}\lambda_4^2 + \frac{1}{3}Q_{4555}\lambda_4 + 2Q_{2456},$

(-14) $Q_{1666} = -\frac{1}{2}\lambda_4 Q_{3566} + Q_{3366} - \frac{1}{2}Q_{3445},$

(-14) $Q_{2446} = \frac{8}{3}\wp_{14} - 2\wp_{22} + \frac{8}{3}\wp_{24}\lambda_4 - \frac{2}{3}\wp_{66}\lambda_4\lambda_3 + \frac{5}{6}Q_{3366} + Q_{5566}\lambda_3$
$\qquad\qquad - \frac{1}{6}Q_{5566}\lambda_4^2 + 2\wp_{44}\lambda_3 - \frac{1}{6}\lambda_4 Q_{4455} - \frac{4}{3}\lambda_4 Q_{3566} - \frac{7}{6}Q_{3445},$

(-14) $Q_{2455} = \frac{8}{3}\wp_{14} - 2\wp_{22} + \frac{8}{3}\wp_{24}\lambda_4 - \frac{2}{3}\wp_{66}\lambda_4\lambda_3 - \frac{5}{3}Q_{3366} - Q_{5566}\lambda_3$
$\qquad\qquad + \frac{1}{3}Q_{5566}\lambda_4^2 + 6\wp_{66}\lambda_2 - 2\wp_{44}\lambda_3 + \frac{1}{3}\lambda_4 Q_{4455} + \frac{2}{3}\lambda_4 Q_{3566} + \frac{1}{3}Q_{3445},$

(-15) $Q_{1566} = -\frac{1}{4}Q_{2445} - \frac{1}{4}Q_{2455}\lambda_4 + \frac{3}{4}\wp_{2366} - \frac{3}{2}\wp_{23}\wp_{66} - 3\wp_{26}\wp_{36}$
$\qquad\qquad - \frac{1}{2}\wp_{36}\lambda_3 + \frac{3}{2}\wp_{56}\lambda_2,$

(-15) $Q_{3356} = -\frac{1}{2}Q_{2445} + 3\wp_{36}\lambda_3 + 3\wp_{56}\lambda_2 - \frac{1}{2}Q_{2566}\lambda_4 - \frac{1}{2}Q_{2366},$

(-15) $Q_{3444} = -\frac{3}{4}Q_{2445} + 4\wp_{36}\lambda_4^2 - \frac{3}{2}Q_{3556}\lambda_4 + \frac{13}{2}\wp_{36}\lambda_3 + \frac{9}{2}\wp_{56}\lambda_2$
$\qquad\qquad - \frac{3}{4}Q_{2566}\lambda_4 - \frac{15}{4}Q_{2366},$

(-16) $Q_{3346} = -2Q_{1466} + 4\lambda_1 - Q_{1556} - Q_{2266} + 16\wp_{16}\lambda_4 + 3\wp_{35}\lambda_3$
$\qquad\qquad - \wp_{55}\lambda_2 + 2\wp_{26}\lambda_3 - 2Q_{2356},$

(-16) $Q_{3355} = 8Q_{1466} - 8\lambda_1 + 6Q_{1556} - 32\wp_{16}\lambda_4 + 4\wp_{35}\lambda_3 - 2Q_{2356},$

(-16) $Q_{2444} = 2\lambda_4\lambda_2 + 10\lambda_1 + 36\wp_{16}\lambda_4 + \frac{3}{2}\wp_{35}\lambda_3 + \frac{15}{2}\wp_{55}\lambda_2 - Q_{3456}\lambda_4$
$\qquad\qquad - \frac{3}{2}Q_{2556}\lambda_4 - 7\wp_{26}\lambda_3 - 6\wp_{46}\lambda_2 + 3\wp_{33}\lambda_4 + 6\lambda_4\wp_{3456}$
$\qquad\qquad - 12\lambda_4\wp_{34}\wp_{56} - 12\lambda_4\wp_{35}\wp_{46} - 12\lambda_4\wp_{36}\wp_{45} + 4\wp_{26}\lambda_4^2$
$\qquad\qquad - 6\wp_{35}\lambda_4^2 - 9Q_{1466} - 9Q_{1556} + \frac{3}{2}Q_{2266} + 3Q_{2356},$

(-17) $Q_{1456} = -2\wp_{13} - \wp_{45}\lambda_2 + 2\wp_{25}\wp_{26} + \wp_{22}\wp_{56} + 4\wp_{15}\lambda_4 - Q_{2346}$
$\qquad\qquad - \frac{1}{12}Q_{4555}\lambda_3 - \frac{1}{2}\wp_{2256} + \wp_{25}\lambda_3 + \frac{1}{3}\wp_{45}\lambda_4\lambda_3 + \frac{4}{3}\wp_{34}\lambda_3,$

(-17) $Q_{1555} = 4\wp_{13} + 6\wp_{45}\lambda_2 + 3Q_{2256} - 8\wp_{15}\lambda_4 + 6Q_{2346} + \frac{1}{2}Q_{4555}\lambda_3$
$\qquad\qquad - 6\wp_{25}\lambda_3 - 2\wp_{45}\lambda_4\lambda_3 - 8\wp_{34}\lambda_3,$

(-17) $Q_{2355} = -4\wp_{13} - 4\wp_{45}\lambda_2 - 2Q_{2256} + 8\wp_{15}\lambda_4 - 2Q_{2346} + 4\wp_{25}\lambda_3 + 4\wp_{34}\lambda_3,$

(-17) $Q_{3345} = -2\wp_{25}\lambda_3 + \frac{4}{3}\wp_{34}\lambda_3 - \frac{2}{3}\wp_{45}\lambda_4\lambda_3 + 2\wp_{45}\lambda_2 + Q_{2256} + \frac{1}{6}Q_{4555}\lambda_3,$

(-18) $Q_{1446} = -6\wp_{12} + 6\wp_{66}\lambda_1 + \frac{1}{2}Q_{3566}\lambda_4^2 - Q_{1455} + 8\wp_{14}\lambda_4 - Q_{3566}\lambda_3$
$\qquad\qquad - Q_{2345} - \frac{1}{2}Q_{3344} - \frac{1}{2}Q_{3366}\lambda_4 + \frac{1}{2}Q_{3445}\lambda_4,$

(-18) $Q_{2246} = 2\wp_{44}\lambda_2 - 4\wp_{66}\lambda_1 - 2\wp_{66}\lambda_4\lambda_2 + \frac{1}{2}Q_{3566}\lambda_3 + Q_{3366}\lambda_4$
$\qquad\qquad - Q_{3445}\lambda_4 - Q_{3566}\lambda_4^2 + 2\wp_{24}\lambda_3 + \frac{3}{2}Q_{5566}\lambda_2 + Q_{1455},$

(-18) $Q_{2255} = -8\wp_{12} - 8\wp_{44}\lambda_2 + 16\wp_{66}\lambda_1 + \frac{32}{3}\wp_{14}\lambda_4 + 8\wp_{66}\lambda_4\lambda_2$

$\qquad\qquad - \frac{2}{3}Q_{3566}\lambda_3 - \frac{8}{3}Q_{3366}\lambda_4 + \frac{4}{3}Q_{3445}\lambda_4 + \frac{4}{3}Q_{3566}\lambda_4^2,$

$\qquad\qquad + 4\wp_{23}\wp_{45} - \frac{2}{3}\lambda_4\lambda_3\wp_{55}\wp_{66} - \frac{4}{3}\lambda_4\lambda_3\wp_{56}^2 - \frac{8}{3}\wp_{66}\lambda_3^2$

$\qquad\qquad + \frac{1}{3}\lambda_4\lambda_3\wp_{5566} + 4\wp_{24}\wp_{35} + 4\wp_{25}\wp_{34} - 3Q_{5566}\lambda_2$

$\qquad\qquad - 2\wp_{2345} + \frac{1}{3}Q_{4455}\lambda_3 + \frac{8}{3}\wp_{24}\lambda_3 - 2Q_{1455},$

(-19) $Q_{1366} = -\frac{3}{10}Q_{2245} - \frac{2}{5}Q_{2344} - \frac{3}{10}Q_{2566}\lambda_3 - \frac{1}{4}Q_{2366}\lambda_4 + \frac{3}{20}Q_{2455}\lambda_4$

$\qquad\qquad + \frac{3}{20}Q_{2566}\lambda_4^2 + \frac{12}{5}\wp_{56}\lambda_1 - \frac{3}{10}\wp_{56}\lambda_4\lambda_2 + \frac{7}{10}\wp_{36}\lambda_4\lambda_3 + \frac{9}{5}\wp_{36}\lambda_2 - \frac{1}{4}Q_{3556}\lambda_3,$

(-19) $Q_{3336} = -\frac{3}{5}Q_{2245} + \frac{6}{5}Q_{2344} - \frac{3}{5}Q_{2566}\lambda_3 + \frac{3}{2}Q_{2366}\lambda_4 + \frac{3}{10}Q_{2455}\lambda_4$

$\qquad\qquad + \frac{3}{10}Q_{2566}\lambda_4^2 + \frac{38}{5}\wp_{36}\lambda_2 - \frac{16}{5}\wp_{56}\lambda_1 - \frac{3}{5}\wp_{56}\lambda_4\lambda_2 - \frac{33}{5}\wp_{36}\lambda_4\lambda_3 + \frac{3}{2}Q_{3556}\lambda_3,$

(-19) $Q_{1445} = 8\wp_{56}\lambda_1 - \frac{1}{2}Q_{2245} - Q_{2344} - \frac{1}{2}Q_{2566}\lambda_3 - \frac{3}{2}Q_{2366}\lambda_4$

$\qquad\qquad + \frac{1}{2}Q_{2566}\lambda_4^2 + \frac{1}{2}Q_{2455}\lambda_4 - 2\wp_{56}\lambda_4\lambda_2 + 3\wp_{36}\lambda_4\lambda_3 + 3\wp_{36}\lambda_2 - \frac{3}{4}Q_{3556}\lambda_3,$

(-20) $Q_{2244} = +2\lambda_3\lambda_2 + 28\lambda_4\lambda_1 + 2\wp_{55}\lambda_4\lambda_2 - 6\wp_{35}\lambda_4\lambda_3 + 3\wp_{33}\lambda_3$

$\qquad\qquad - 24\wp_{46}\lambda_1 + 112\wp_{16}\lambda_4^2 - 44\wp_{16}\lambda_3 + 28\wp_{55}\lambda_1 + 9\wp_{35}\lambda_2$

$\qquad\qquad + 20Q_{1356} - 6Q_{1444} + 10Q_{1266} - 22Q_{1466}\lambda_4 + Q_{2266}\lambda_4$

$\qquad\qquad + 2Q_{2356}\lambda_4 + 2Q_{3456}\lambda_3 - 22Q_{1556}\lambda_4 - Q_{2556}\lambda_3 - 6\wp_{26}\lambda_2,$

(-20) $Q_{2336} = 8\wp_{16}\lambda_3 - 2\wp_{55}\lambda_1 + 2\wp_{35}\lambda_2 - 2Q_{1356} - 2Q_{1266},$

(-20) $Q_{3335} = -32\lambda_4\lambda_1 + 24\wp_{46}\lambda_1 - 128\wp_{16}\lambda_4^2 + 56\wp_{16}\lambda_3 - 32\wp_{55}\lambda_1$

$\qquad\qquad - 30Q_{1356} + 8Q_{1444} - 12Q_{1266} + 24Q_{1466}\lambda_4 + 24Q_{1556}\lambda_4 + 4\wp_{35}\lambda_2,$

(-21) $Q_{1256} = -\frac{1}{2}Q_{2236} - Q_{1346} - \frac{1}{12}Q_{4555}\lambda_2 + \frac{1}{3}\wp_{45}\lambda_4\lambda_2 + 3\wp_{15}\lambda_3$

$\qquad\qquad + \frac{4}{3}\wp_{34}\lambda_2 - 2\wp_{45}\lambda_1,$

(-21) $Q_{1355} = \frac{1}{2}Q_{2236} + Q_{1346} + \frac{1}{4}Q_{4555}\lambda_2 - \frac{1}{2}Q_{2335} - \wp_{45}\lambda_4\lambda_2 + \wp_{15}\lambda_3,$

(-21) $Q_{3334} = -\frac{3}{2}Q_{2236} - 3Q_{1346} + \frac{1}{4}Q_{4555}\lambda_2 + \frac{3}{2}Q_{2335} - \wp_{45}\lambda_4\lambda_2$

$\qquad\qquad + 9\wp_{15}\lambda_3 + 12\wp_{34}\lambda_2 - 8\wp_{45}\lambda_1,$

(-22) $Q_{1345} = 5\wp_{66}\lambda_0 - \wp_{11} + \frac{16}{15}\wp_{14}\lambda_4^2 + \frac{11}{5}\wp_{66}\lambda_4\lambda_1 - \frac{7}{30}Q_{3566}\lambda_4\lambda_3$

$\qquad\qquad - \frac{1}{5}Q_{3344}\lambda_4 - \frac{1}{2}Q_{1255} + \frac{10}{3}\wp_{14}\lambda_3 - \wp_{44}\lambda_1 - Q_{1246}$

$\qquad\qquad + \frac{2}{15}Q_{3566}\lambda_4^3 - \frac{1}{15}Q_{3366}\lambda_4^2 - \frac{1}{3}Q_{3366}\lambda_3 + \frac{1}{6}Q_{3445}\lambda_3$

$\qquad\qquad - \frac{2}{5}Q_{2345}\lambda_4 - \frac{4}{5}\wp_{12}\lambda_4 + \frac{2}{15}Q_{3445}\lambda_4^2,$

(-22) $Q_{2226} = -6\wp_{11} + \frac{7}{2}Q_{3566}\lambda_2 + \frac{1}{2}Q_{4455}\lambda_2 + 6Q_{5566}\lambda_1 + \frac{32}{5}\wp_{14}\lambda_4^2$

$\qquad\qquad - \frac{14}{5}\wp_{66}\lambda_4\lambda_1 - \frac{39}{10}Q_{3566}\lambda_4\lambda_3 - \frac{24}{5}\wp_{12}\lambda_4 + 4\wp_{24}\lambda_2$

$\qquad\qquad + 12\wp_{14}\lambda_3 + 6\wp_{44}\lambda_1 - 2\wp_{66}\lambda_3\lambda_2 - 6Q_{1246} + \frac{4}{5}Q_{3445}\lambda_4^2$

$\qquad\qquad - \frac{6}{5}Q_{3344}\lambda_4 - \frac{2}{5}Q_{3366}\lambda_4^2 + \frac{3}{2}Q_{3366}\lambda_3 - \frac{3}{2}Q_{3445}\lambda_3$

$\qquad\qquad - 10\wp_{66}\lambda_0 - \frac{12}{5}Q_{2345}\lambda_4 + \frac{4}{5}Q_{3566}\lambda_4^3 + \frac{1}{2}Q_{5566}\lambda_4\lambda_2,$

(-22) $Q_{2235} = -2\wp_{11} - Q_{3566}\lambda_2 - 4Q_{5566}\lambda_1 - \frac{32}{15}\wp_{14}\lambda_4^2 + \frac{38}{5}\wp_{66}\lambda_4\lambda_1$

$\qquad\qquad + \frac{22}{15}Q_{3566}\lambda_4\lambda_3 + \frac{8}{15}\wp_{12}\lambda_4 + 10\wp_{66}\lambda_0 - Q_{1255} + \frac{2}{15}Q_{3366}\lambda_4^2$

$\qquad\qquad - 10\wp_{44}\lambda_1 + 2Q_{1246} - \frac{4}{15}Q_{3445}\lambda_4^2 + \frac{2}{5}Q_{3344}\lambda_4 + \frac{4}{3}\wp_{14}\lambda_3$

$\qquad\qquad - \frac{4}{3}Q_{3366}\lambda_3 + \frac{2}{3}Q_{3445}\lambda_3 + \frac{4}{5}Q_{2345}\lambda_4 - \frac{4}{15}Q_{3566}\lambda_4^3,$

$$(-22) \quad Q_{2334} = -4\wp_{11} - Q_{3566}\lambda_2 + 2Q_{5566}\lambda_1 - \tfrac{32}{15}\wp_{14}\lambda_4{}^2 - \tfrac{32}{5}\wp_{66}\lambda_4\lambda_1$$
$$+ \tfrac{7}{15}Q_{3566}\lambda_4\lambda_3 + \tfrac{8}{5}\wp_{12}\lambda_4 + Q_{1255} + \tfrac{4}{3}\wp_{14}\lambda_3 + 4\wp_{44}\lambda_1$$
$$- \tfrac{4}{15}Q_{3445}\lambda_4{}^2 + \tfrac{2}{5}Q_{3344}\lambda_4 + \tfrac{2}{15}Q_{3366}\lambda_4{}^2 + \tfrac{2}{3}Q_{3366}\lambda_3$$
$$- \tfrac{1}{3}Q_{3445}\lambda_3 + \tfrac{4}{5}Q_{2345}\lambda_4 - \tfrac{4}{15}Q_{3566}\lambda_4{}^3.$$

Appendix D. The two-term addition formula

The Abelian functions associated with C satisfy the following two-term addition formula:

$$-\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = f(\boldsymbol{u},\boldsymbol{v}) - f(\boldsymbol{v},\boldsymbol{u}),$$

where $f(\boldsymbol{u},\boldsymbol{v})$ is a finite polynomial of Abelian functions. We write $f(\boldsymbol{u},\boldsymbol{v})$ as

$$f(\boldsymbol{u},\boldsymbol{v}) = P_{30} + P_{26} + P_{22} + P_{18} + P_{14} + P_{10} + P_6 + P_2,$$

where each P_k contains the terms with weight $-k$ in the Abelian functions and weight $k-30$ in $\boldsymbol{\lambda}$:

$$P_{30} = \tfrac{1}{4}Q_{114466}(\boldsymbol{u}) + \tfrac{5}{3}Q_{3566}(\boldsymbol{v})Q_{1356}(\boldsymbol{u}) - \tfrac{1}{3}\wp_{14}(\boldsymbol{v})Q_{2356}(\boldsymbol{u})$$
$$+ \tfrac{1}{3}Q_{1556}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) - \tfrac{1}{4}Q_{2236}(\boldsymbol{u})\wp_{25}(\boldsymbol{v}) - \tfrac{2}{5}Q_{2345}(\boldsymbol{u})\wp_{33}(\boldsymbol{v})$$
$$+ \tfrac{1}{10}Q_{3344}(\boldsymbol{v})Q_{3456}(\boldsymbol{u}) - \tfrac{1}{2}\wp_{25}(\boldsymbol{v})Q_{1346}(\boldsymbol{u}) + \tfrac{1}{2}Q_{4556}(\boldsymbol{v})\wp_{11}(\boldsymbol{u})$$
$$+ \tfrac{3}{5}Q_{3344}(\boldsymbol{u})\wp_{16}(\boldsymbol{v}) + \tfrac{1}{24}Q_{1346}(\boldsymbol{v})Q_{4555}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1145}(\boldsymbol{v})\wp_{56}(\boldsymbol{u})$$
$$+ \tfrac{1}{12}Q_{2236}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1444}(\boldsymbol{v})Q_{3566}(\boldsymbol{u}) - \wp_{46}(\boldsymbol{v})Q_{1146}(\boldsymbol{u})$$
$$- \tfrac{1}{2}Q_{1556}(\boldsymbol{v})\wp_{22}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1155}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{5}{6}Q_{3566}(\boldsymbol{v})Q_{1266}(\boldsymbol{u})$$
$$+ \tfrac{7}{5}\wp_{12}(\boldsymbol{v})Q_{3456}(\boldsymbol{u}) + \wp_{26}(\boldsymbol{v})Q_{1246}(\boldsymbol{u}) + Q_{2346}(\boldsymbol{v})\wp_{15}(\boldsymbol{u})$$
$$+ \tfrac{1}{10}Q_{2345}(\boldsymbol{u})Q_{2556}(\boldsymbol{v}) - \tfrac{2}{3}Q_{1356}(\boldsymbol{v})Q_{4455}(\boldsymbol{u}) - \tfrac{1}{20}Q_{2245}(\boldsymbol{v})Q_{2566}(\boldsymbol{u})$$
$$+ \wp_{36}(\boldsymbol{v})Q_{1245}(\boldsymbol{u}) - \tfrac{1}{5}\wp_{33}(\boldsymbol{v})Q_{3344}(\boldsymbol{u}) + Q_{1246}(\boldsymbol{v})\wp_{35}(\boldsymbol{u})$$
$$+ \tfrac{1}{3}Q_{1466}(\boldsymbol{u})Q_{3445}(\boldsymbol{v}) - \tfrac{4}{3}Q_{1466}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) - \wp_{26}(\boldsymbol{v})\wp_{11}(\boldsymbol{u})$$
$$+ \tfrac{1}{5}\wp_{16}(\boldsymbol{v})Q_{2345}(\boldsymbol{u}) - \tfrac{1}{12}Q_{3445}(\boldsymbol{v})Q_{2266}(\boldsymbol{u}) + \tfrac{1}{2}Q_{5566}(\boldsymbol{v})Q_{1244}(\boldsymbol{u})$$
$$+ \tfrac{1}{6}Q_{1466}(\boldsymbol{v})Q_{3366}(\boldsymbol{u}) + \tfrac{1}{4}8Q_{4555}(\boldsymbol{v})Q_{2335}(\boldsymbol{u}) + \tfrac{1}{3}Q_{4455}(\boldsymbol{v})Q_{1266}(\boldsymbol{u})$$
$$- \tfrac{1}{6}Q_{4455}(\boldsymbol{v})Q_{1444}(\boldsymbol{u}) + \tfrac{1}{5}Q_{2345}(\boldsymbol{v})Q_{3456}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1156}(\boldsymbol{v})\wp_{45}(\boldsymbol{u})$$
$$+ Q_{2256}(\boldsymbol{v})\wp_{15}(\boldsymbol{u}) - \tfrac{32}{5}\wp_{12}(\boldsymbol{v})\wp_{16}(\boldsymbol{u}) - \tfrac{7}{3}Q_{1356}(\boldsymbol{v})\wp_{24}(\boldsymbol{u})$$
$$- \tfrac{1}{2}\wp_{33}(\boldsymbol{v})Q_{1455}(\boldsymbol{u}) + \tfrac{1}{2}\wp_{44}(\boldsymbol{v})Q_{1166}(\boldsymbol{u}) - Q_{1455}(\boldsymbol{u})\wp_{16}(\boldsymbol{u})$$
$$- \tfrac{1}{48}Q_{2236}(\boldsymbol{v})Q_{4555}(\boldsymbol{u}) - \tfrac{1}{20}Q_{3344}(\boldsymbol{v})Q_{2556}(\boldsymbol{u}) - \tfrac{1}{10}Q_{2566}(\boldsymbol{v})Q_{2344}(\boldsymbol{u})$$
$$- \tfrac{1}{10}Q_{3556}(\boldsymbol{v})Q_{2245}(\boldsymbol{u}) - \tfrac{1}{20}Q_{3556}(\boldsymbol{v})Q_{2344}(\boldsymbol{u}) + \tfrac{1}{8}Q_{2445}(\boldsymbol{v})Q_{2366}(\boldsymbol{u})$$
$$+ \tfrac{1}{6}Q_{1556}(\boldsymbol{v})Q_{3366}(\boldsymbol{u}) - \tfrac{1}{2}Q_{1255}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - \tfrac{1}{2}Q_{1144}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})$$
$$- \tfrac{5}{12}Q_{2335}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) + \tfrac{1}{6}Q_{3366}(\boldsymbol{v})Q_{2266}(\boldsymbol{u}) + \tfrac{2}{3}\wp_{24}(\boldsymbol{v})Q_{1266}(\boldsymbol{u})$$
$$- \tfrac{1}{2}Q_{2256}(\boldsymbol{v})\wp_{23}(\boldsymbol{u}) - \wp_{22}(\boldsymbol{u})Q_{1466}(\boldsymbol{v}) + \tfrac{1}{2}\wp_{55}(\boldsymbol{v})Q_{1146}(\boldsymbol{u})$$
$$+ \tfrac{1}{6}Q_{1346}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) - \tfrac{1}{3}\wp_{14}(\boldsymbol{u})Q_{2266}(\boldsymbol{v}) + \wp_{13}(\boldsymbol{v})Q_{2456}(\boldsymbol{u})$$
$$- \tfrac{1}{5}\wp_{12}(\boldsymbol{v})Q_{2556}(\boldsymbol{u}) - \tfrac{2}{3}\wp_{24}(\boldsymbol{v})Q_{1444}(\boldsymbol{u}) + \tfrac{1}{4}Q_{2335}(\boldsymbol{v})\wp_{25}(\boldsymbol{u})$$
$$+ \wp_{44}(\boldsymbol{v})Q_{1244}(\boldsymbol{u}) + \tfrac{14}{5}\wp_{33}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) - Q_{2346}(\boldsymbol{v})\wp_{23}(\boldsymbol{u}) - \wp_{35}(\boldsymbol{v})\wp_{11}(\boldsymbol{u})$$
$$- \tfrac{1}{3}Q_{2356}(\boldsymbol{v})Q_{3366}(\boldsymbol{u}) + \tfrac{1}{3}Q_{3445}(\boldsymbol{v})Q_{1556}(\boldsymbol{u}) - \tfrac{1}{6}Q_{3445}(\boldsymbol{v})Q_{2356}(\boldsymbol{u}),$$

$$
\begin{aligned}
P_{26} = \Big[& \tfrac{1}{4} Q_{1155}(u) - Q_{1146}(u) - \tfrac{1}{60} Q_{2556}(u) Q_{3366}(v) - \tfrac{1}{2} Q_{1556}(u) Q_{4455}(v) \\
& + \tfrac{1}{24} Q_{3566}(u) Q_{2266}(v) - \tfrac{7}{6} Q_{5566}(v) Q_{1266}(u) + \tfrac{1}{2} Q_{1466}(v) Q_{4455}(u) \\
& - \tfrac{1}{15} Q_{3445}(v) Q_{3456}(u) - \tfrac{7}{3} Q_{5566}(v) Q_{1356}(u) + \tfrac{7}{6} Q_{3566}(u) Q_{1556}(v) \\
& + \tfrac{1}{30} Q_{2556}(u) Q_{3445}(v) - \tfrac{1}{2} Q_{5566}(u) Q_{1444}(v) + \tfrac{1}{20} Q_{3556}(v) Q_{2445}(u) \\
& - \tfrac{1}{30} Q_{3456}(v) Q_{3366}(u) + \tfrac{1}{40} Q_{2566}(u) Q_{2445}(v) + \tfrac{7}{6} Q_{1466}(v) Q_{3566}(u) \\
& + \tfrac{1}{6} Q_{3566}(u) Q_{2356}(v) + 2 Q_{1466}(v) \wp_{24}(u) - \tfrac{3}{10} Q_{3445}(v) \wp_{33}(u) \\
& + \tfrac{4}{15} Q_{2556}(u) \wp_{14}(v) - \tfrac{1}{6} \wp_{45}(u) Q_{1346}(v) - \tfrac{128}{15} \wp_{16}(v) \wp_{14}(u) \\
& + \tfrac{2}{5} \wp_{33}(u) Q_{3366}(v) - \tfrac{8}{15} Q_{3366}(v) \wp_{16}(u) - \tfrac{1}{24} Q_{2335}(u) \wp_{45}(v) \\
& + 2 \wp_{23}(u) \wp_{15}(v) - \wp_{11}(u) \wp_{55}(v) + 4 \wp_{16}(v) \wp_{22}(u) - \tfrac{12}{5} \wp_{33}(u) \wp_{14}(v) \\
& + 3 \wp_{44}(u) Q_{1266}(v) + \tfrac{4}{3} \wp_{44}(v) Q_{1444}(u) + \tfrac{3}{5} \wp_{36}(v) Q_{2344}(u) \\
& - 2 \wp_{35}(u) \wp_{12}(v) + \tfrac{1}{24} \wp_{45}(u) Q_{2236}(v) - \tfrac{8}{15} Q_{3456}(u) \wp_{14}(v) \\
& - \wp_{66}(v) Q_{1244}(u) - 2 \wp_{25}(u) \wp_{13}(v) - \tfrac{1}{5} \wp_{36}(u) Q_{2245}(v) \\
& + \tfrac{14}{15} \wp_{16}(v) Q_{3445}(u) + 6 \wp_{44}(u) Q_{1356}(v) - 2 Q_{1556}(u) \wp_{24}(v) \Big] \lambda_4,
\end{aligned}
$$

$$
\begin{aligned}
P_{22} = \Big[& \wp_{35}(v) \wp_{14}(u) - Q_{1466}(v) Q_{5566}(u) - \tfrac{58}{15} \wp_{16}(u) Q_{3566}(v) \\
& - Q_{1556}(u) Q_{5566}(v) - \tfrac{1}{40} Q_{2566}(u) Q_{3556}(v) - \tfrac{1}{5} Q_{3456}(u) Q_{3566}(v) \\
& - 2 Q_{1466}(u) \wp_{44}(v) - \tfrac{1}{2} Q_{3366}(v) \wp_{35}(u) + 2 \wp_{44}(u) Q_{1556}(v) \\
& + \tfrac{1}{2} \wp_{15}(v) \wp_{25}(u) - \tfrac{1}{24} \wp_{23}(u) Q_{4555}(v) + \tfrac{2}{5} \wp_{33}(u) Q_{3566}(v) \\
& + \tfrac{8}{3} Q_{1266}(u) \wp_{66}(v) + \tfrac{16}{3} Q_{1356}(v) \wp_{66}(u) - \tfrac{4}{3} \wp_{23}(v) \wp_{34}(u) \\
& + \tfrac{1}{10} Q_{2556}(u) Q_{3566}(v) + \tfrac{4}{3} Q_{1444}(u) \wp_{66}(v) + \tfrac{1}{4} Q_{3445}(v) \wp_{35}(u) \\
& - \tfrac{4}{3} \wp_{16}(u) Q_{4455}(v) - 4 \wp_{16}(u) \wp_{24}(v) - \tfrac{1}{4} Q_{2366}(u) \wp_{36}(v) \\
& - \tfrac{13}{6} \wp_{15}(u) \wp_{34}(v) + \tfrac{1}{24} \wp_{15}(u) Q_{4555}(v) - \wp_{23}(v) \wp_{25}(u) \Big] \lambda_3 \\
& + \Big[-6 Q_{1356}(v) \wp_{66}(u) - \tfrac{1}{15} Q_{3456}(u) Q_{3566}(v) - 4 \wp_{44}(u) Q_{1556}(v) \\
& - \tfrac{3}{2} Q_{1466}(v) Q_{5566}(u) + 3 Q_{1266}(u) \wp_{66}(v) + \tfrac{1}{2} Q_{2366}(u) \wp_{36}(v) \\
& - \tfrac{1}{10} Q_{2445}(u) \wp_{36}(v) + 4 Q_{1466}(u) \wp_{44}(v) - \tfrac{3}{10} \wp_{33}(u) Q_{3566}(v) \\
& + \tfrac{1}{30} Q_{2556}(u) Q_{3566}(v) - \tfrac{4}{3} Q_{1444}(u) \wp_{66}(v) + \tfrac{32}{3} \wp_{16}(u) \wp_{24}(v) \\
& + \tfrac{8}{3} \wp_{16}(u) Q_{4455}(v) + \tfrac{106}{15} \wp_{16}(u) Q_{3566}(v) + \tfrac{1}{20} Q_{2566}(u) Q_{3556}(v) \\
& + \tfrac{3}{2} Q_{1556}(u) Q_{5566}(v) \Big] \lambda_4^2,
\end{aligned}
$$

$$
\begin{aligned}
P_{18} = \Big[& \tfrac{3}{10} Q_{2566}(u) \wp_{36}(v) - \wp_{23}(u) \wp_{45}(v) - \tfrac{1}{5} \wp_{2345}(v) - \tfrac{1}{10} Q_{3344}(v) \\
& + \tfrac{2}{5} \wp_{12}(u) + \tfrac{3}{4} \wp_{56}(u) Q_{2366}(v) + \tfrac{1}{24} Q_{4555}(u) \wp_{25}(v) + \tfrac{4}{3} \wp_{34}(u) \wp_{25}(v) \\
& - \tfrac{1}{24} \wp_{34}(v) Q_{4555}(u) - 2 \wp_{45}(v) \wp_{15}(u) - \tfrac{1}{24} \wp_{55}(u) Q_{3445}(v) \\
& + \tfrac{1}{3} \wp_{55}(u) \wp_{14}(v) + \tfrac{1}{6} \wp_{55}(u) Q_{3366}(v) - \tfrac{3}{5} Q_{3556}(u) \wp_{36}(v) \Big] \lambda_2 \\
& + \Big[\tfrac{7}{20} Q_{2566}(u) \wp_{36}(v) - \tfrac{3}{5} Q_{3556}(u) \wp_{36}(v) - \tfrac{1}{4} Q_{3566}(u) \wp_{35}(v) \\
& - \tfrac{2}{3} \wp_{16}(v) Q_{5566}(u) - \tfrac{1}{6} \wp_{45}(u) \wp_{15}(v) + \tfrac{1}{3} \wp_{23}(u) \wp_{45}(v) \\
& + 2 \wp_{66}(u) Q_{1556}(v) + 2 \wp_{66}(u) Q_{1466}(v) + \tfrac{4}{3} \wp_{44}(v) \wp_{16}(u) \Big] \lambda_4 \lambda_3 \\
& + \Big[4 \wp_{66}(u) Q_{1556}(v) + \tfrac{1}{10} Q_{2566}(u) \wp_{36}(v) + 8 \wp_{16}(v) Q_{5566}(u) \\
& + 4 \wp_{66}(u) Q_{1466}(v) - \tfrac{64}{3} \wp_{44}(v) \wp_{16}(u) \Big] \lambda_4^3,
\end{aligned}
$$

$$P_{14} = \Big[\wp_{25}(u)\wp_{45}(v) + \wp_{22}(u) + \tfrac{1}{3}Q_{3445}(u) + \tfrac{4}{3}\wp_{14}(v) + \tfrac{5}{12}Q_{3366}(v)$$
$$- \tfrac{11}{20}Q_{3556}(u)\wp_{56}(v) - 6\wp_{44}(u)\wp_{26}(v) + 3Q_{5566}(v)\wp_{26}(u)$$
$$+ \tfrac{18}{5}\wp_{16}(u)\wp_{66}(v) - \tfrac{1}{2}Q_{4455}(v)\wp_{46}(u) + \tfrac{23}{12}Q_{3566}(v)\wp_{55}(u)$$
$$- \tfrac{1}{24}Q_{4555}(u)\wp_{45}(v) - \tfrac{27}{10}\wp_{33}(u)\wp_{66}(v) - \tfrac{7}{3}\wp_{24}(u)\wp_{55}(v)$$
$$- \tfrac{3}{5}Q_{3456}(u)\wp_{66}(v) - 2\wp_{24}(v)\wp_{46}(u) + \tfrac{2}{3}Q_{4455}(v)\wp_{55}(u)$$
$$- 3\wp_{35}(u)Q_{5566}(v) - \tfrac{2}{3}\wp_{34}(u)\wp_{45}(v) - \tfrac{11}{2}\wp_{35}(u)\wp_{44}(v)$$
$$+ \tfrac{3}{10}Q_{2556}(u)\wp_{66}(v) - \tfrac{3}{2}Q_{3566}(v)\wp_{46}(u) + \tfrac{1}{10}Q_{2566}(v)\wp_{56}(u)\Big]\lambda_1$$
$$+ \Big[\tfrac{1}{24}Q_{3566}(v)\wp_{55}(u) + \tfrac{1}{6}\wp_{25}(u)\wp_{45}(v) - \tfrac{1}{10}Q_{3556}(u)\wp_{56}(v)$$
$$+ \tfrac{1}{6}\wp_{34}(u)\wp_{45}(v) - \tfrac{1}{20}Q_{2566}(v)\wp_{56}(u) - \tfrac{1}{30}Q_{3366}(v) - \tfrac{1}{15}Q_{3445}(u)$$
$$+ \tfrac{8}{15}\wp_{14}(v)\Big]\lambda_4\lambda_2 - \tfrac{64}{3}\wp_{16}(v)\wp_{66}(u)\lambda_4^4 + \tfrac{68}{3}\wp_{16}(v)\wp_{66}(u)\lambda_4^2\lambda_3$$
$$- \tfrac{32}{3}\wp_{16}(v)\wp_{66}(u)\lambda_3^2,$$

$$P_{10} = \Big[-2\wp_{24}(v) - \tfrac{5}{3}Q_{3566}(v) - \tfrac{5}{2}\wp_{44}(v)\wp_{55}(u) + 5\wp_{36}(u)\wp_{56}(v)$$
$$- \tfrac{1}{24}Q_{4455}(v) - \tfrac{5}{2}\wp_{35}(u)\wp_{66}(v) + \tfrac{5}{4}Q_{5566}(u)\wp_{55}(v)\Big]\lambda_0 + \Big[4\wp_{44}(v)\wp_{46}(u)$$
$$+ \tfrac{8}{3}\wp_{24}(v) + \tfrac{18}{5}\wp_{36}(u)\wp_{56}(v) + 6\wp_{26}(v)\wp_{66}(u) - \tfrac{11}{2}\wp_{35}(u)\wp_{66}(v)$$
$$+ 6\wp_{44}(v)\wp_{55}(u) - \tfrac{3}{2}Q_{5566}(u)\wp_{46}(v) + \tfrac{7}{3}Q_{5566}(u)\wp_{55}(v) - \tfrac{5}{3}Q_{3566}(u)$$
$$- \tfrac{2}{3}Q_{4455}(u)\Big]\lambda_4\lambda_1,$$

$$P_6 = \tfrac{16}{3}\wp_{66}(u)\wp_{55}(v)\lambda_3\lambda_1 + \tfrac{7}{12}Q_{5566}(v)\lambda_4\lambda_0 - 4\wp_{46}(u)\wp_{66}(v)\lambda_4^2\lambda_1$$
$$- 4\wp_{46}(v)\wp_{66}(u)\lambda_3\lambda_1 + \tfrac{5}{2}\wp_{55}(u)\wp_{66}(v)\lambda_4\lambda_0 - 6\wp_{55}(v)\wp_{66}(u)\lambda_4^2\lambda_1$$
$$- \tfrac{3}{2}Q_{5566}(v)\lambda_3\lambda_1 + \tfrac{4}{3}\wp_{44}(v)\lambda_4\lambda_0 - \tfrac{16}{3}\wp_{44}(v)\lambda_4^2\lambda_1 - 2Q_{5566}(v)\lambda_4^2\lambda_1$$
$$+ 3\wp_{44}(u)\lambda_3\lambda_1,$$

$$P_2 = \tfrac{7}{3}\wp_{66}(u)\lambda_4\lambda_3\lambda_1 - \tfrac{3}{5}\wp_{66}(u)\lambda_2\lambda_1 - \tfrac{16}{3}\wp_{66}(u)\lambda_4^3\lambda_1 + \tfrac{4}{3}\wp_{66}(u)\lambda_4^2\lambda_0 - \tfrac{2}{3}\wp_{66}(u)\lambda_3\lambda_0.$$

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